Midterm Practice - Solutions

Monday, February 19, 2024 7:50 PM

① A necessary and sufficient condition for a tour on D is

$$\forall v \in V(D) : d^-(v) = d^+(v)$$

→ From the given degree sequences, this condition could be satisfied on some Hypothetical D

$\Rightarrow$ a tour is possible □

② G has cycle $\Rightarrow |V(G)| \leq |E(G)|$

via strong induction on $|E(G)|$

Basis $P(1) = 8$ is the only possible example

Assume we have $P(n)$ w/ cycle

$P(k) = P(n) \cdot e$ ← edge contraction

As edge contraction will retain cycles and won't disconnect $P(k)$, we have via I.H.

$|V(P(k))| \leq |E(P(k))|$

via I.H.

$$|V(P(k))| \leq |E(P(k))|$$

when we re-expand the edge e

edge, we note

$$|V(P(n))| = |V(P(k))| + 1$$

$$|E(P(n))| = |E(P(k))| + 1$$

$$\Rightarrow |V(P(n))| \leq |E(P(n))| \checkmark$$

$|V(G)| \leq |E(G)| \Rightarrow G$ has a cycle

consider some spanning tree of G

→ this tree T has

$$|V(T)| = |V(G)|$$

$$|E(T)| = |V(T)| - 1$$

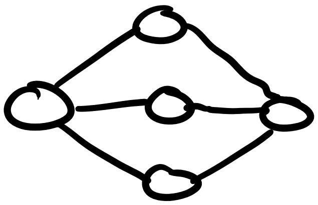
$$|E(G)| \geq |V(G)| > |E(T)|$$

→ so G has strictly more edges than T, and since we've shown T is maximally acyclic

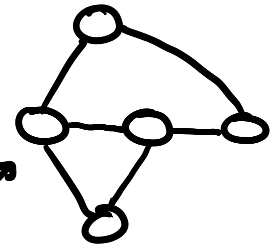
$\Rightarrow$ G must have a cycle □

③ **counter-example**

$S = \{3\ 3\ 2\ 2\ 2\}$

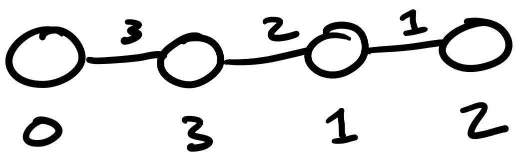


Havel-Hakimi will *always* generate →



Note: largest degree vertices will always connect via HH□

④



✓

⑤ a) There are $\frac{n(n-1)}{2}$ possible unique edges and n possible unique loops

→ the existence of each is a binary choice

$\Rightarrow 2^{\left[\frac{n(n+1)}{2} + n\right]}$ loopy graphs

b) For our edges, we now have

b) For our edges, we now have a ternary choice → no edge, 1 edge, or 2 edges

we still have same for loops

$$\Rightarrow \boxed{3^{\frac{n(n+1)}{2}} 2^n} \text{ loopy multigraphs}$$

⑥ We have proven correctness of Kruskal's MST algorithm

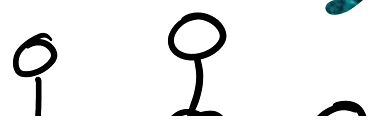
→ This algorithm considers a sorted order of edges

→ with unique edge weights, this order is strict, so only a single output is possible

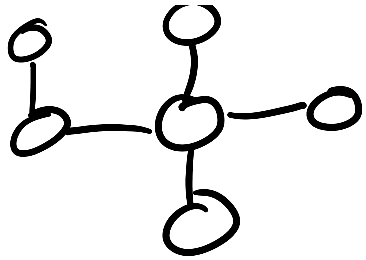
⇒ as this output is guaranteed to be an MST, only this unique output is possible □

⑦

counter-example



|E(G)| = even



$|E(G)| = \text{even}$

but fails Tutte's

$\Rightarrow$ no P.M. $\square$

(8) $\forall v \in V(G): G - v$ has a P.M. $\Leftrightarrow$

$|V(G)| = \text{odd}$ and $\forall S \subseteq V(G): o(G-S) \leq |S|$

($\Rightarrow$) since we require $|V(G)| = \text{even}$ for P.M.

$\rightarrow |V(G-v)| = \text{even}, |V(G)| = \text{odd}$

Define $G' = G - v$

By Tutte, we know

$\forall S' \subseteq V(G'): o(G' - S') \leq |S'|$

Define $S = S' + v$

$\rightarrow G - S = G' - S'$

$o(G-S) = o(G'-S)$

$|S'| = |S| - 1$

$\Rightarrow o(G-S) = o(G'-S') \leq |S'| = |S| - 1$

so $o(G-S) \leq |S|$ ✓

($\Leftarrow$) Define as before:

$G' = G - v$

$S = S' + v, \; S' \subseteq V(G')$

Note: $|V(G)| = \text{odd} \rightarrow |S|$ and $o(G-S)$ have different parity

$\rightarrow o(G-S) \leq |S| - 1$

$\rightarrow o(G'-S') = o(G-S) \leq |S| - 1 = |S'|$

$\Rightarrow$ and since Tutte's holds for G', it has a P.M. □

⑨ By now, you might have realized

{something about degrees}

+ {something about P.M.}

= {degree sum formula} + {Tutte}

Consider some $S \subseteq V(G)$

$G - S \rightarrow H_1, H_2 \ldots H_k$ odd components of $G - S$

Max edges from S to all H_i

Max edges from S to all H_i
$= 3|S|$

Min edges from all H_i to S
$= 2\, o(G-S)$
↑ no single edge cut

However: As each $|V(H_i)| = $ odd

$$\sum_{v \in V(H_i)} d(v) = 3|V(H_i)| - 2 = \text{odd}$$

↑ internal degrees ↑ external degrees

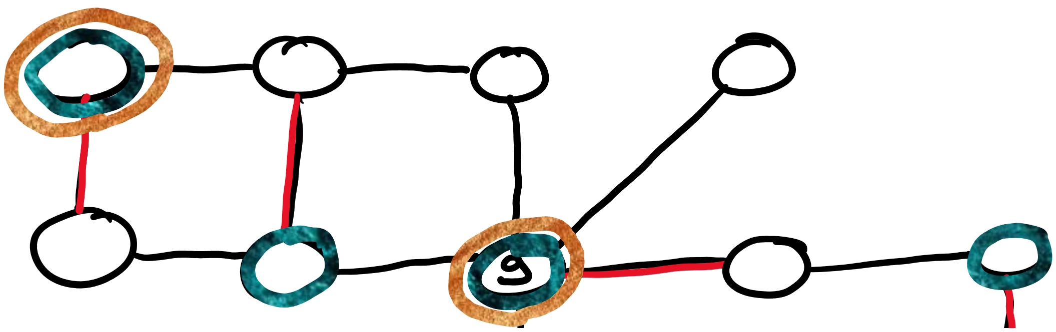
→ so minimum cut must be 3

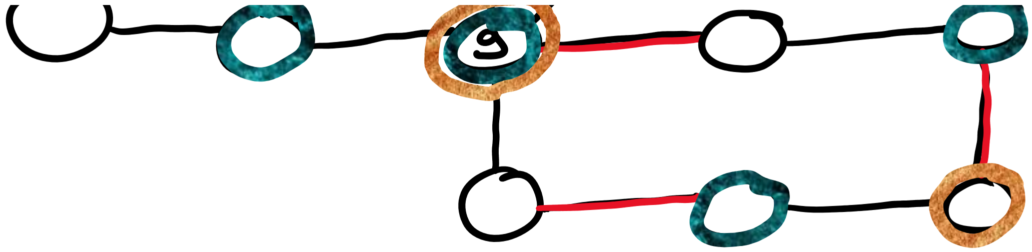
Min edges from H_i to S
$= 3\, o(G-S)$

→ $3\, o(G-S) \leq$ actual cut $\leq 3|S|$

$\Rightarrow o(G-S) \leq |S|$, so we have a P.M. via Tutte □

(10)





a) (red edge) = match

define $S = \{g\} \rightarrow o(G-S) = 3 > |S|$

$\Rightarrow$ no P.M., so match

$|M| = \frac{|V(G)|}{2} - 1$ is optimal □

b) (teal circle) = cover

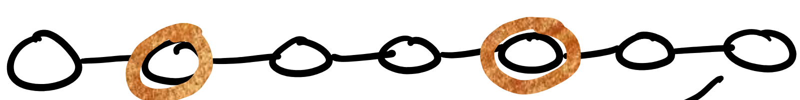
(no odd cycles)

As G is bipartite, we know via König-Egeváry that

$5 = \text{Min cover} = \text{max match} = 5$

c) The diameter of G is 6

Consider u, v-path where $\underbrace{\max_{u,v \in V(G)} d(u,v)}_{\text{max } u,v \text{ distance}}$



we require a dominating set of at least $\left\lceil \frac{|V(P)|}{3} \right\rceil = \left\lceil \frac{7}{3} \right\rceil = 3$

to simply cover the longest

to simply cover the longest shortest path P in G

$\Rightarrow$ since our set is of cardinality 3, we achieve this lower bound $\square$