Midterm - Solutions

Friday, February 23, 2024 3:18 PM

① Weakly connected digraph D has closed trail containing $\forall e \in E(D)$

$$\iff$$

$$\forall v \in V(D): d^+(v) = d^-(v)$$

($\Rightarrow$) Consider any $v \in V(D)$

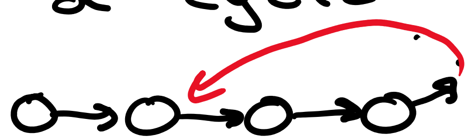
→ along the trail every time we reach v via an in-edge, we must exit via an out edge

→ this holds for all vertices and all edges per the assumption

$\Rightarrow \forall v \in V(D), d^+(v) = d^-(v)$

($\Leftarrow$) we note all $d^+(v) = d^-(v) > 0$ as D is weakly connected

→ From class, we proved via a maximal path argument that D must have a cycle



PROOF BY ALGORITHM

we will construct a closed trail via

duction

via Induction on $E(D)$

Basis: $P(1) \rightarrow \emptyset$ only case, trivial trail

Consider some $P(n)$

$\rightarrow$ we know $\exists C \subseteq P(n)$

(proven in class via extremal argument, we can guarantee $\forall v \in V(D): d^-(v) = d^+(v) \geq 1$ as D is weakly connected)

$P(k) = P(n) - C$

I.H. on components of $P(k)$ gives us trails $T_1 \ldots T_m$

Our Algorithm:

Start on some $v \in V(C)$

Traverse along C, add edges to T

If we reach some v with $e \in N(v): e \in T_i$

Add close T_i to T

Repeat until all T_i are in T and we reach original v

and we reach original v
output closed T with
all $e \in E(D)$ $\square$

(2) We'll prove via weak induction
on the sum of the sequence

Basis $P(0) \to$ trivially empty graph

Assume via I.H. $P(k)$ is realizable

We'll show for $P(k+2)$

Case 1: $+1$ is added to two separate $s_i \, s_j \in P(k)$
→ add an edge to graph realized by $P(k)$ between vertices i, j represented by s_i, s_j

Case 2: $+2$ is added to a single s_i
→ add a loop to i

Case 3: $+1$ added to $s_i \in S$ and a new s_j is added to S
→ add new leaf j with edge to i

Case 4: Case 1 but both s_i, s_j new
→ add edge between new i, j

→ add edge between new i, j

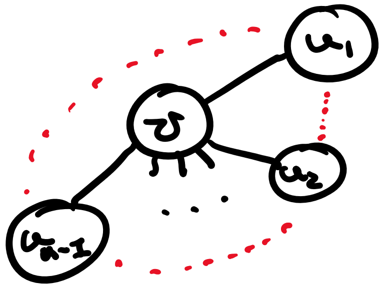
Case 5: Case 2 but new s_i

→ add new i with self loop

⇒ We have shown $P(k+2)$ is realizable □

(Strong induction or ALGORITHM are probably much simpler proofs)

③ Dominating set of size 1 implies that G has max diameter of 2



Note: a star graph with with n vertices is smallest graph

↳ The number of configurations is solely a function of edges between $u_i, u_j \in N(v)$ (and selection of v)

→ Consider induced subgraph on $N(v)$

→ Note: for an n-vertex graph, we have $2^{\frac{n(n-1)}{2}}$

⇒ n-1 vertices gives us, plus n choices for v

$\boxed{n\left[2^{\frac{(n-1)(n-2)}{2}}\right]}$ possible graphs □

④ $v_1 \quad v_2 \quad v_3 \quad v_4$

$[\tfrac{1}{2} \quad 0 \quad 0 \quad 0]$

(4)

$$A = \begin{array}{c|c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array} \qquad D^{-1} = \begin{bmatrix} 1/3 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

$$M^T = \begin{bmatrix} 1/3 & 0 & 1/3 & 1/3 \\ 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix} \qquad M = \begin{bmatrix} 1/3 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix}$$

(5) Strong induction on $E(T)$

Basis: $P(1) =$ o–o trivially K_2

Consider tree $P(n)$

$P(k) = P(n) - L$, where L is *all* leaves in $P(n)$

Note: eccentricities for all $v \in V(P(k))$ will be reduced by exactly 1, as longest shortest paths will end at a leaf

→ The center of $P(k)$ is the same as the center of $P(n)$

I.H. on $P(k)$ gives us the

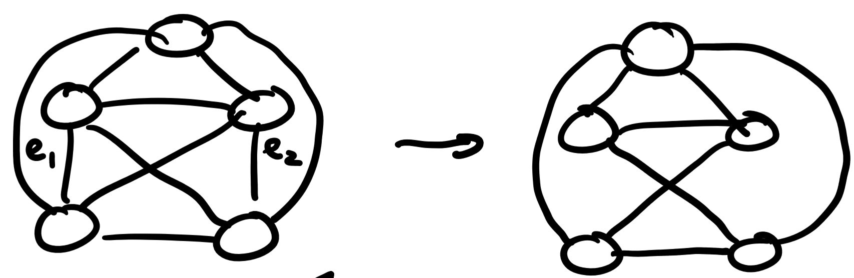
I.H. on $P(k)$ gives us the center being K_1 or K_2

we add back L, all eccentricities increase by exactly 1 for $P(n)$

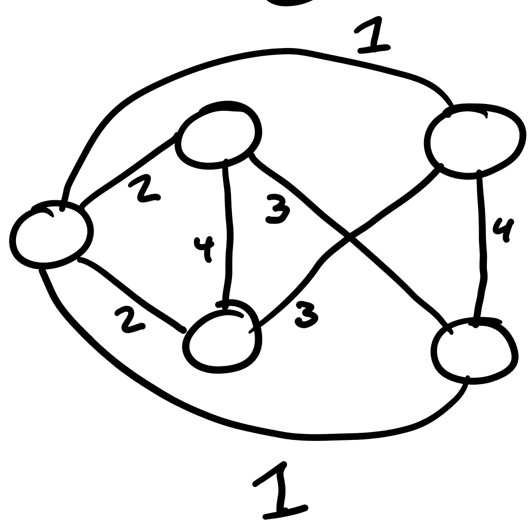
$\Rightarrow$ The center will remain the same as $P(k)$, K_1 or K_2 $\square$

6

K_5



G



We consider an MST algo like Kruskal, and add edges in non-decreasing order, adding a new v with each weight

$\rightarrow$ edges with weights $\{1, 1, 2, 2\}$ will always be in any MST $\square$

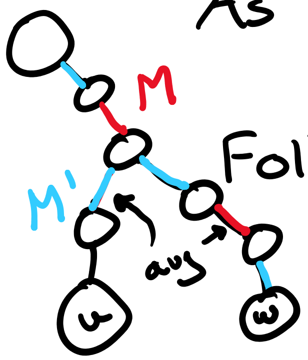
Note: don't need to show work

7 Consider M' as a larger match $|M'| = |M| + 1$ created with aug path P, $M' = P \Delta M$

there now exist aug. path P' on M'

created with aug path P, ...

Can there now exist aug. path P' on M' that starts at u? Assume yes



As P' needed $P \Delta M$ to exist, P and P' must intersect

Follow P' from u to P as M'-alt path, then follow P in direction of M-alt path to its endpoint w

→ This gives us an M-aug path from u to w *contradiction*

=> As this holds for any M, u will remain unsaturated □

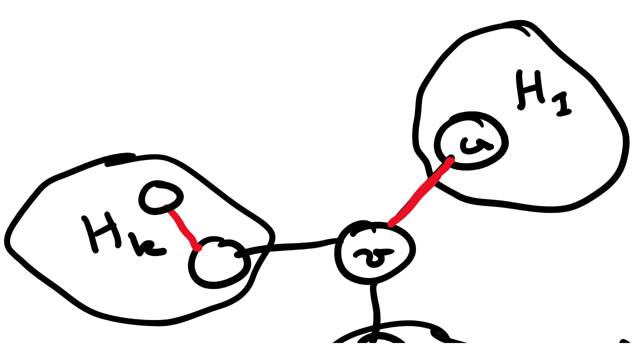
8. We note $|V(T)|$ = even

Consider some $v \in V(T)$

Case 1: $T-v$ is connected

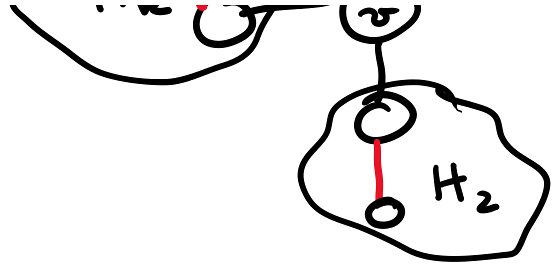
$|V(T-v)| = |V(T)| - 1 = \text{odd}$

Case 2: $T-v$ is disconnected into some k comps $H_1 \dots H_k$



Define H_1 as the comp. with v's matched neighbor of u

...

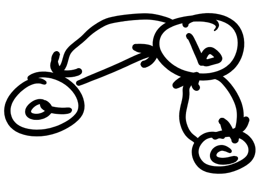


of u

Note: all other H_i with have P.M.s, as the deletion of v does not affect their matches

$\Rightarrow$ All H_i have even order, except for H_1, as all $w \in V(H_1)$ have a matched partner except for the unmatched u $\square$

(9) Trivial disproof



$$A = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

(hopefully you didn't overthink this)

(10) At each iteration all men propose and all women reject, where the order of proposals and order of rejections is irrelevant

$\Rightarrow$ Hence, the output will always be the same, as there is no point at which the algorithm

be the same, as there is
no point at which the algorithm
can diverge on a given iteration □

$\boxed{\text{No}}$ - consider counter-example

$x: a > b$ $\quad$ $a: y > x$

$y: b > a$ $\quad$ $b: x > y$

Either match $\{(x, a), (y, b)\}$
$\{(x, b), (y, a)\}$
is stable □