Weekly Problems 10
Sunday, March 31, 2024 5:37 PM

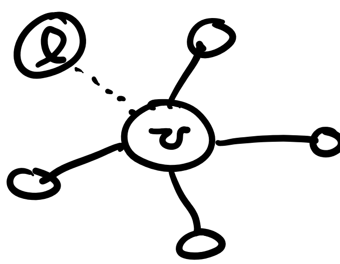
① We'll show via weak induction on $|V|$

Basis $P(1) = 0$ → trivially only outer face

Assume via I.H. for some $P(k)$ we have only one outer face

→ we create $P(k+1)$ by adding leaf ℓ as neighbor of some v

Consider v with embedding $P(k)$



→ $N(v)$ are all aligned around v

→ the only way to add ℓ is between two of $N(v)$

=> This gives an embedding of $P(k+1)$ with only a single face

=> This applies to any $P(k)$ and any v and ℓ, hence it is valid for any tree □

② Induction on $|C|$, where C is a

(c) Induction on $|C|$, where C is a set of cycles in G

Basis $P(0)$: proven above for trees

Assume we have some $P(n)$ graph with $n = |C|$

→ We construct $P(k)$ by selecting some $e \in C_i : C_i \in C$ and deleting it

via I.H. we have:

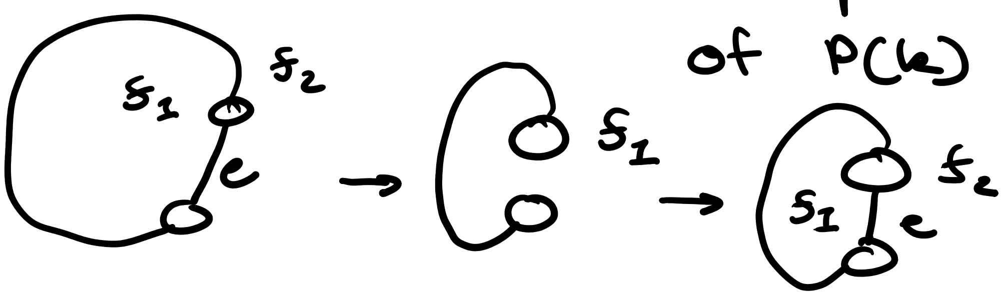
$$n_k - e_k + f_k = 2 \quad \text{on} \quad P(k)$$

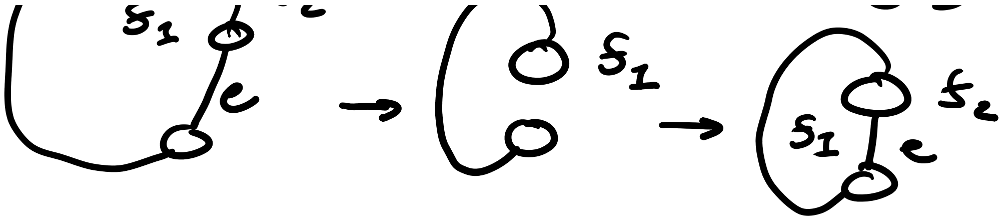
we note on $P(n)$

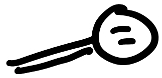
$n_n = n_k$ as we don't reduce $|V|$

$e_n = e_k + 1$ as we deleted a single edge

$f_n = f_k + 1$ as adding back e splits one face of $P(k)$ into 2





 and 

$$n_k - e_k + f_k = 2$$

$$n_n - (e_n - \cancel{1}) + (f_n - \cancel{1}) = 2$$

$$n_n - e_n + f_n = 2 \quad \square$$