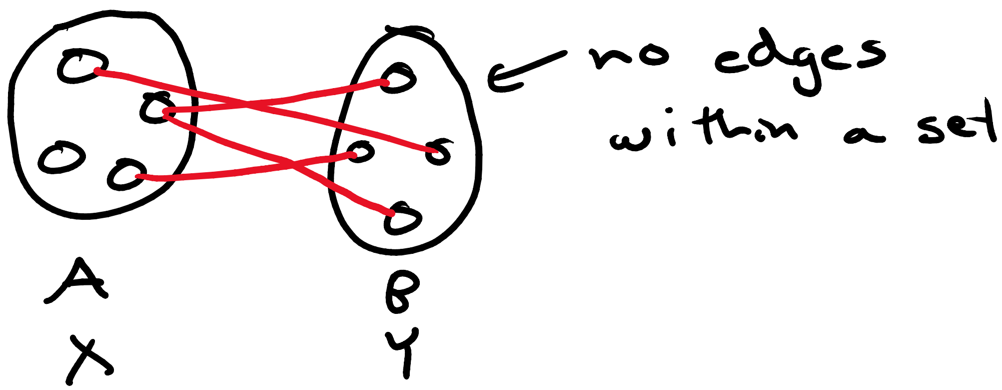
Lecture 2 - Graph Classes and Isomorphism

Thursday, January 11, 2024 8:27 AM

Fun Fact of the Day: In grad school, there was only a single class
that Slota's advisor did not allow him to take.
That class: graph theory

Bipartite graphs

Graphs whose vertex set can be separated into two vertex-disjoint sets, s.t. (such that) between any two vertices in one of these sets, there are no edges

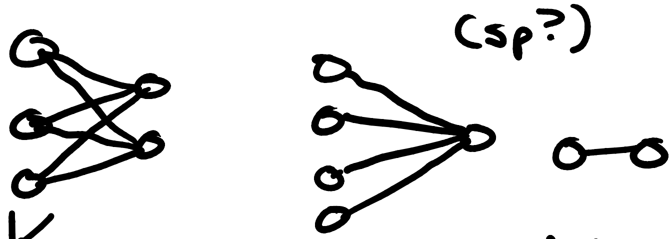


biclique

a.k.a. complete bipartite graph

a bipartite graph with sets X, Y with all $x \in X$ connecting to all $y \in Y$ and vice-versa

$y \in Y$ and vice-versa
(sp?)



$K_{3,2}$ $K_{4,1}$ $K_{1,1}$

Graph Representation

→ adjacency matrix representation

matrix A: an $n \times n$ matrix representing the structure of some graph G, where $n = |V(G)|$

if edge (i,j) exists $\in E(G)$
→ $a_{ij} \in A$ is nonzero

(assume vertices are labeled or sorted from $1 \ldots n$)

For undirected simple graphs:

$a_{ij} \in \{0, 1\}$

zeros on the diagonal

$$a_{ij} = a_{ji}$$

For multigraphs:

$a_{ij} \geq 0$ → value is number of (i, j) edges

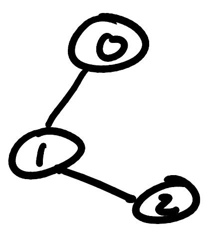
For loopy graphs:

$a_{ii} \in \{0, 2\}$ → value of 2 indicates self loop on vertex i

For loopy multigraphs:

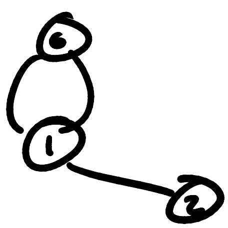
$a_{ii} \geq 0$ → value is 2x the number of loops on vertex i

$a_{ij} \geq 0$



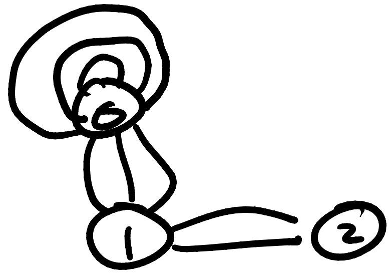
$$\begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$\sum row_i = d(i)$$



$$\begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$\sum row_1 = 3 = d(1)$$

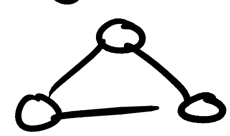


$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & & & \\ 1 & & \left[\begin{matrix} 6 & 3 & 0 \\ 3 & 0 & 2 \\ 0 & 2 & 0 \end{matrix}\right] & \\ 2 & & & \end{array}$$

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 6 & 3 & 0 \\ 1 & 3 & 0 & 2 \\ 2 & 0 & 2 & 0 \end{array}$$

Graph Properties and Classes

Property: a description or characteristic that same graph holds

Consider K_3: 

Property: $|V(K_3)| = |E(K_3)| = 3$

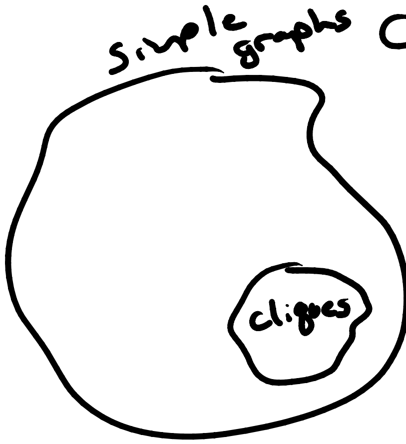
Property: K_3 is cyclic (has cycles)

Property: $\forall v \in V(K_3): d(v) = 2$
aka K_3 is <u>2-regular</u>

Graph class: a set of all possible graphs having the same set of properties

Examples:

simple graphs: no multi-edges
no self-loops



clique graphs: simple
all possible edges exist

Note: graph classes can have a subset | superset relationship

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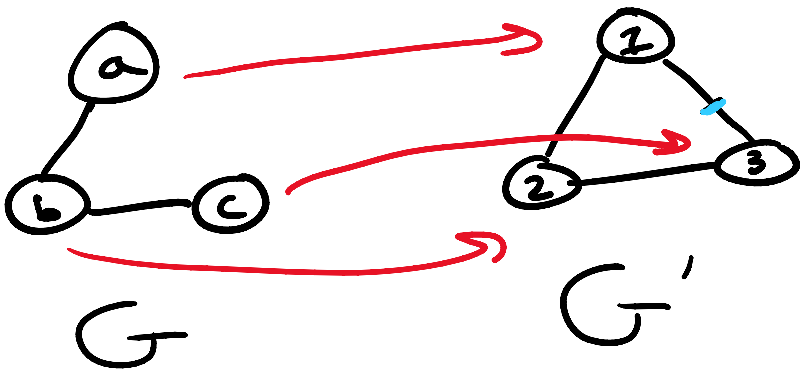
Graph Isomorphism

Graph $G = \{V, E\}$ is <u>isomorphic</u> to $G' = \{V', E'\}$ if there exists a bijective mapping using bijective function f

$$f: V \to V'$$

$$\forall v \in V \to f(v) = u \in V'$$

$$\forall e = (a, b) \in E \to (f(a), f(b)) = h \in E'$$



$= h \in E'$

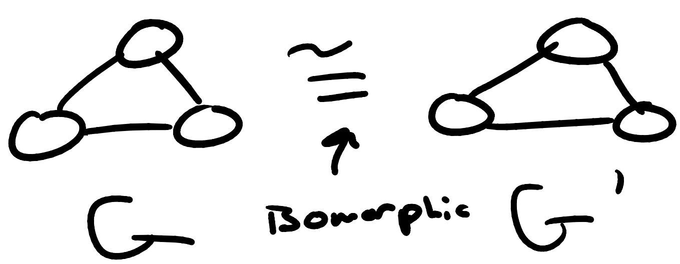
$f(a) = 1$
$f(b) = 2$
$f(c) = 3$

$(a, b) \to (f(a), f(b))$
$(1, 2) \in E'$
$(b, c) \to (f(b), f(c))$
$(2, 3) \in E'$

$\to$ not isomorphic

Recall for bijection:

1. Must have 1-to-1 mapping
2. Functional mapping $f: V \to V'$ must be invertible to some $g: V' \to V$



If $G \cong G'$:

$|V(G)| = |V(G')|$

$|E(G)| = |E(G')|$

list of all degrees for all $v \in V$

sorted degree sequence is going to be equivalent

The diameters will be equal

length of the longest shortest path

The girths will be equal

shortest cycle

Constituent subgraphs are identical

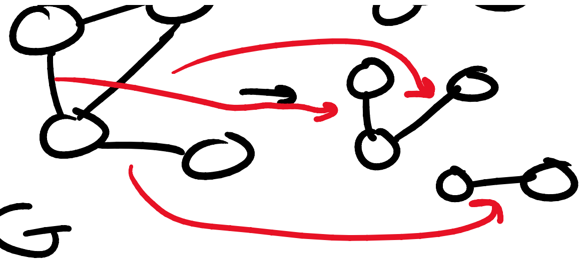
subgraph $H \subseteq G$

$V(H) \subseteq V(G)$

$E(H) \subseteq E(G)$

Likewise, a decomposition on G also exists on G'

deconstructing same G into edge-disjoint



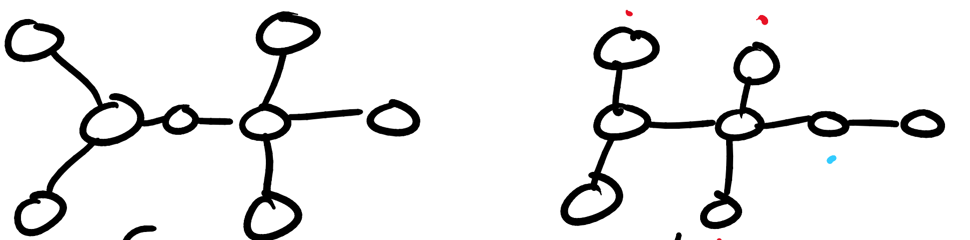
G

G into edge-disjoint subgraphs, using all edges

Note: the properties P are all necessary but not sufficient

→ all G in iso. class must have P

→ any graph with P is in iso. class



G

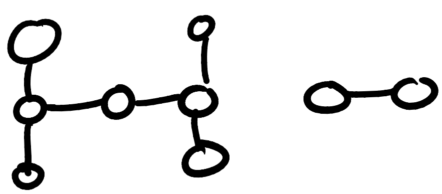
G'

degree sequence

$S = \{1, 1, 1, 1, 1, 2, 3, 4\}$

$S = \{1, 1, 1, 1, 1, 2, 3, 4\}$

decomposition of G → G' has no such decomposition



⇒ G is not isomorphic to G'

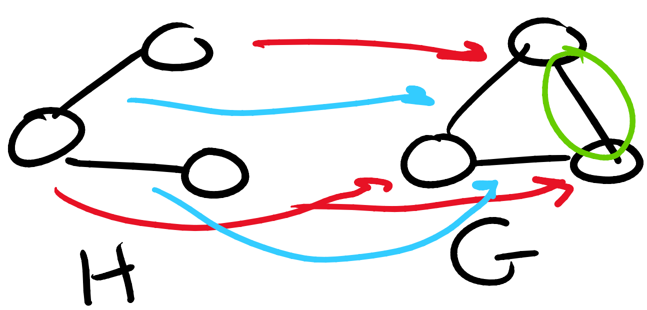
In general: to show an isomorphic

In general to show an isomorphic
relationship doesn't hold, one
just needs to show some property
differs between the two graphs

To show an isomorphic relationship
DOES hold → need to find and
validate that bijective function

↳ usually, much harder to do

<u>subgraph isomorphism</u>: determing
whether some graph H is a
subgraph of some graph G



H is a non-induced subgraph of G

<u>Induced subgraph</u>

↪ consider some vertex set of H and a mapped vertex set of G, all edges among vertices in G exist among all vertices in H