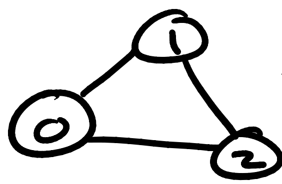
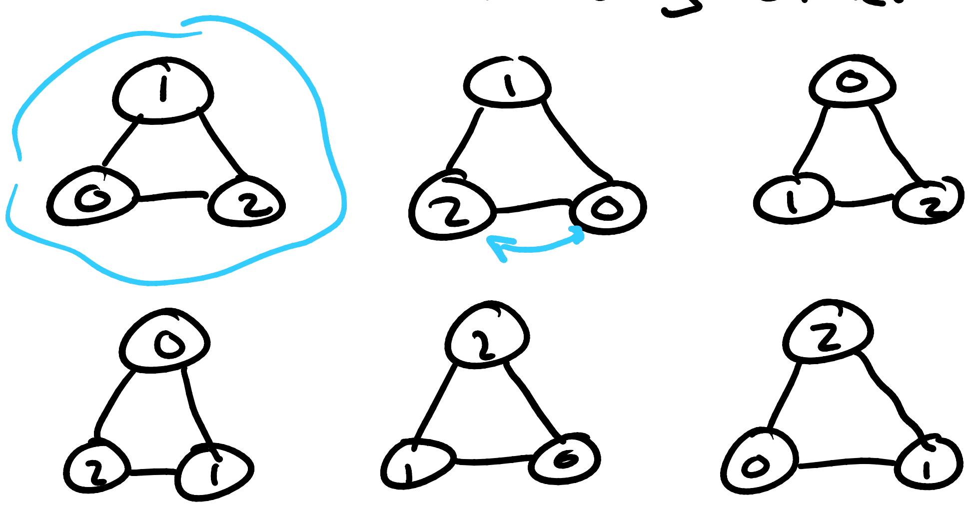
'Graph Theory is a terminological jungle, in which any newcomer may plant a tree.'
- John Barnes

Automorphism

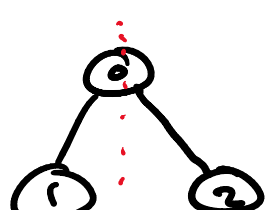
→ An isomorphic mapping of same G to itself (s.t. the $\underbrace{\text{edge list}}$ is preserved)

G ↳  → $\{(1,2),(0,1),(0,2)\}$

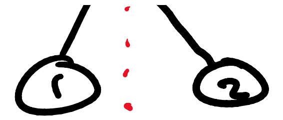
K_n: we can map any vertex to any other vertex



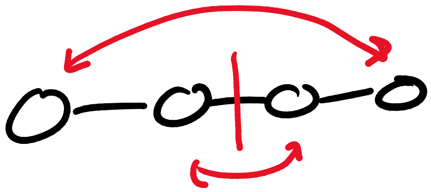
Note: $n!$ total automorphisms

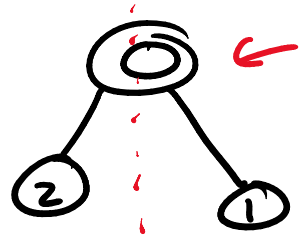
what about:  $E = \{(0,1),(0,2)\}$

what about:



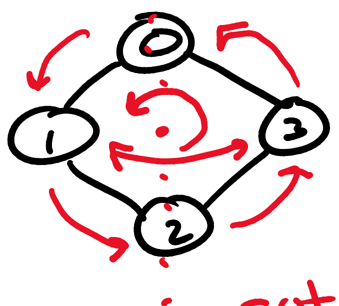
$E = \{(0,1), (0,4)\}$





← symmetric about that axis | vertex

C_4:



$E = \{(0,1), (1,2), (2,3), (0,3)\}$

rotational symmetry too!

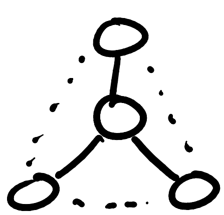
More definition

<u>complement</u> of $G \to \bar{G}$ (assume G is simple)

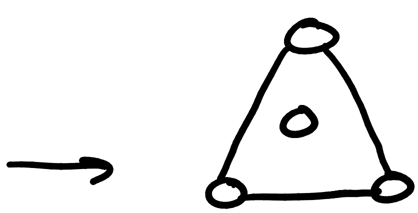
$$V(\bar{G}) = V(G)$$

$$E(\bar{G}) = \{\forall u,v \in V(\bar{G}) : (u,v) \notin E(G)\}$$

not in



G



$\bar{G}$

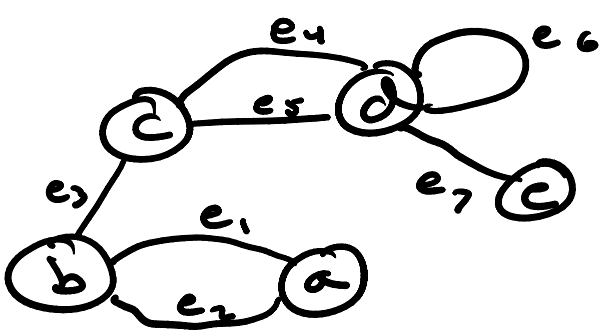
Time to take

Time to take a stroll

walk: a list of vertices and edges s.t. each listing is adjacent/incident to the listing preceeding and proceeding

trail: as above, except edges do not repeat

path: as above, except edges and vertices do not repeat



$W: \{a\, e_1\, b\, e_3\, c\, e_3\, b\, e_3\, c\, e_4\, d\, e_6\, d\, e_7\, e\}$

$w: \{e_1\, e_3\, e_3\, e_3\, e_4\, e_6\, e_7\}$

$T: \{e_1\, e_3\, e_4\, e_6\, e_7\}$

$P: \{e_1\, e_3\, e_4\, e_7\}$

a,e-path

end

start end

A walk/trail/path that starts and ends at the same vertex is <u>closed</u> otherwise, they are <u>open</u>

Note: a closed path is really just a cycle

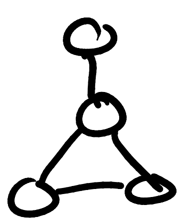
<u>Length</u>: number of edges traversed

<u>Hop</u>: traversal of a single edge

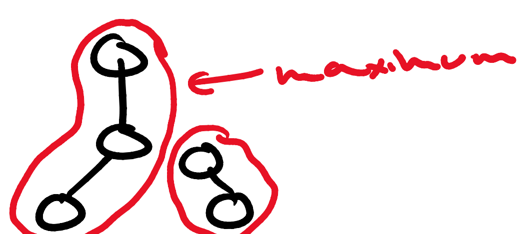
Let's get connected

Recall: G is <u>connected</u> if

$$\forall u,v \in V(G): \exists\, u,v\text{-path}$$



G is connected



G' is disconnected

<u>connected component</u> : a <u>maximal</u> connected subgraph of G

<u>maximal</u>: can not be made larger

<u>maximum</u>: the largest possible

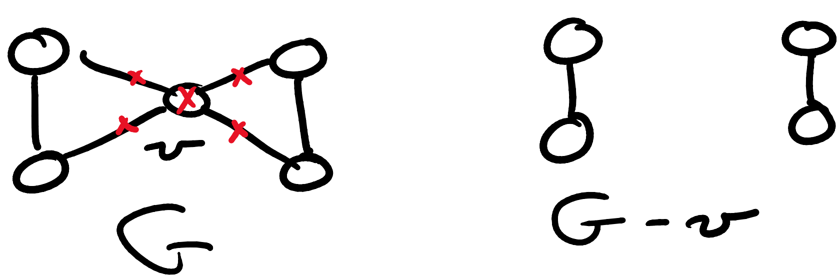
Note: same for <u>minimal</u> | <u>minimum</u>

but smaller | smallest

<u>cut vertex</u>: some $v \in V(G)$ s.t.

$G - v$ has more connected components than G

vertex deletion
→ remove v and all incident edges on v



<u>cut edge</u>: some $e \in E(G)$ s.t.

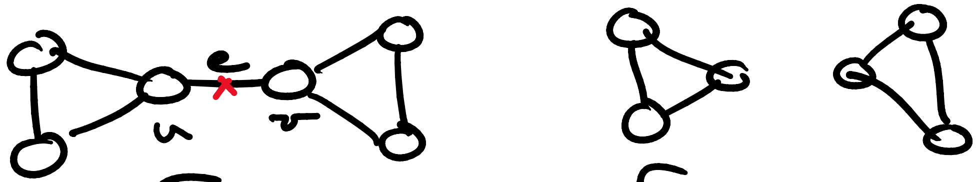
$G - e$ has more components than G

↗ edge deletion

than G

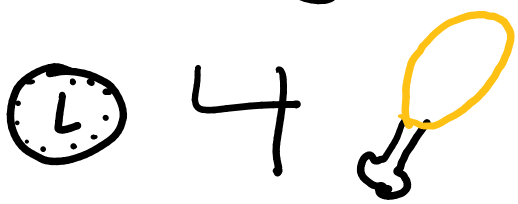
edge deletion

→ just remove e



G G−e

Time for the meat of graph theory



aka
proovin' stuff

weak induction

Prove: $2^1 + 2^2 + 2^3 \ldots + 2^n = 2^{n+1} - 2$

Basis: $P(1) \Rightarrow 2^1 = 2^{1+1} - 2 = 2$ ✓

Inductive step: $P(n = k+1)$

Inductive Hypothesis: we assume what we're trying to prove holds for some $P(k)$

→

Show: P(k+1) holds

$$P(n=k+1) = \underbrace{2^1 + 2^2 + \ldots + 2^k}_{\text{I.H.} \Rightarrow 2^{k+1} - 2} + 2^{k+1}$$

$$P(n=k+1) = 2^{k+1} - 2 + 2^{k+1}$$

$$= 2^{\underline{k+2}} - 2$$

$\rightarrow k+1+1 = n+1$

$$= 2^{n+1} - 2 \quad \square$$

← end of proof aka QED

weak induction

P(1), P(2)... P(k), P(k+1) ... P(∞)

↑ basis

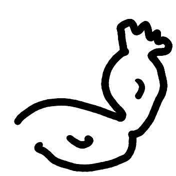
↑ assume holds via I.H.

↑ show it holds

holds for P(1)... P(∞)

Note: basis is smallest that fits assumptions/relation

strong induction



P(1), ... P(k) ... P(n) ... P(∞)

↑ basis

↑ assume holds

↑ show it

basis

assume holds via I.H. for *all* $1 \le k < n$

show it holds

length is odd

Example Proof:

Show every closed odd walk contains an odd cycle

strong Induction on the length of the walk

Basis: P(1):  ✓

Inductive step: $P(n > k \ge 1)$

↳ Assume we have P(n), which is of length n, odd, closed

Consider the cases

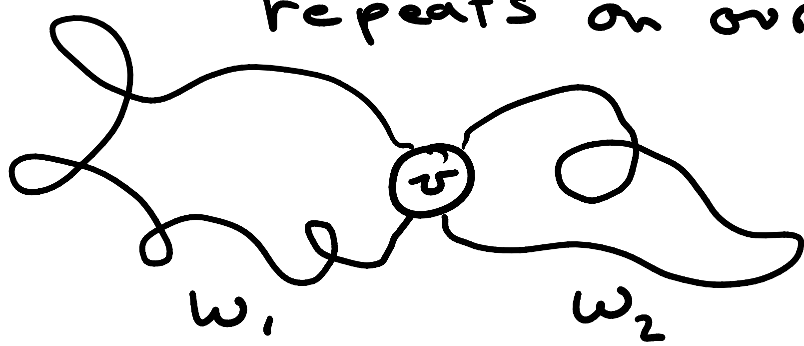
Generally: what possible configurations we can use to simplify our proof

Case 1: no vertices repeat on our walk

=> trivial case, our odd walk

$\Rightarrow$ trivial case, our odd walk is just a cycle ✓

Case 2: at least one vertex v repeats on our walk W



$|W| = |w_1| + |w_2|$

% integer parity %

% % % %

this means odd+odd=even
odd+even=odd
even+even=even

this implies w.l.o.g. $|w_1| = \text{odd}$

without loss of generality

Note: $|w_1| < |W|$

↳ we define $P(k) = w_1$

and use our I.H. on $P(k)$

$\Rightarrow \exists$ some odd cycle on $P(k)$

Bring it back to $P(n)$

↳ show our property still holds on $P(n)$

In this case:

adding w_2 back to w_1

to create $w = P(n)$ will not delete that odd cycle on w $\square$