Lecture 4 - More proof techniques, Eulerian circuits

Sunday, January 21, 2024 5:45 PM

Break it down

new notation $\mathcal{C}$ = class of graphs (s.b)

$\mathcal{C}_1$ = {closed, odd, walk}

$\mathcal{C}$ = {closed, odd, walk, has odd cycle}

really, show if $G \in \mathcal{C}_1$

$\Rightarrow G \in \mathcal{C}_2$

To approach this or any graph proof: consider the properties defining the graph class under consideration

walk: vertices, edges can repeat

closed: start/end at same vertex (or any vertex on the walk)

recall: closed walk w/no repeats → cycle

odd: think: parity

→ We can simplify our arguments by considering

arguments by considering
subclasses of our C_1

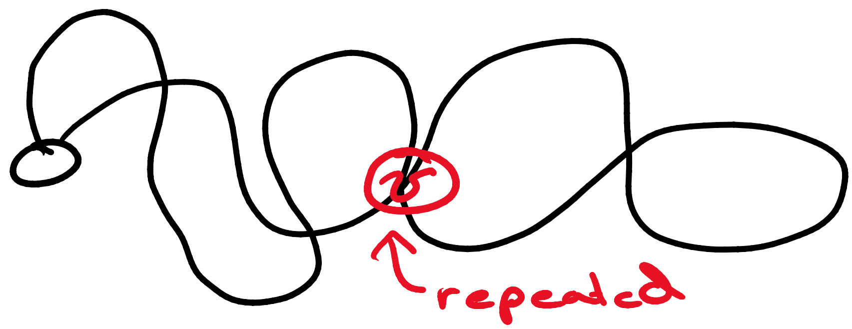
* Consider the cases *

trivial case

1. No repeated vertices
2. Yes repeated vertices

cases must cover all of C_1

↳ consider the structure



Useful: draw it out

Note that we can split our walk

↳ we have $|W_1|$ and $|W_2|$

odd via parity

Note: inductive proofs are inherently
recursive

→ Our arguments must recurse
back to the base case

OR
a trivial subcase

Also: our W_1 produced via our defined construction must be $W_1 \in C_1$

Proof Techniques

* Basic structural arguments
  - basis for most logical statements
* Consider the cases
  - simplify trivial cases
  - simplify structure to consider
* Parity arguments

  odd + odd = even

  even + even = even

  odd + even = odd

* Necessity and sufficiency

  aka: equivalence relations

  $C_A = \{G$ with property $A\}$

$\mathcal{C}_1 = \{G \text{ with property } A\}$

$\mathcal{C}_2 = \{G \text{ with property } B\}$

defines equivalence

$\mathcal{C}_1 \Leftrightarrow \mathcal{C}_2$

class $\mathcal{C}_1$ and $\mathcal{C}_2$ are equivalent

so any $G \in \mathcal{C}_1 \Rightarrow G \in \mathcal{C}_2$

and any $G \in \mathcal{C}_2 \Rightarrow G \in \mathcal{C}_1$

To prove:

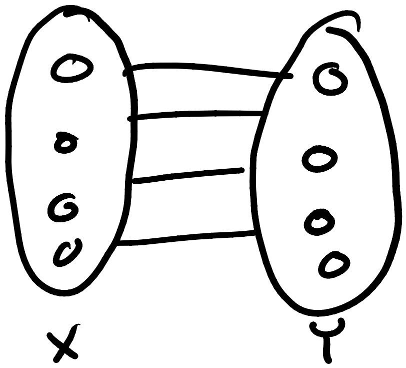
Show property A implies property B

show property B implies property A

Prove: G has no odd cycles $\Leftrightarrow G$ is bipartite

First show: $\Leftarrow$

so G is bipartite $\Rightarrow G$ has no odd cycles



Note: cycle is a closed path

Consider all possible paths from some $x \in X$

We note we always traverse from some $x \in X$ to $y \in Y$ and from some $y \in Y$ to $x \in X$ on a path

$\rightarrow$ any odd path from some $x \in X$ ends at some $y \in Y$

$\Rightarrow$ no closed odd path exists ✓

Now, we show the other direction

G has no odd cycles $\Rightarrow$ G is bipartite

w.l.o.g. assume G is connected

(as we can apply the arguments to any component of G) consider some $v \in V(G)$

define: $f(u)$ where $u \in V(G)$

$f(u)$ = shortest path distance from v

$X : \{\forall x \in V(G) : f(x) = \text{even}\}$

$Y : \{\forall y \in V(G) : f(y) = \text{odd}\}$

Q: are X and Y independent

↳ no edges among vertices in an independent set

Note: $X \cap Y = \emptyset$

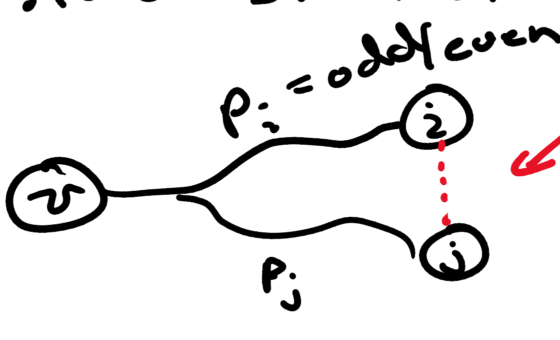
intersection

empty set

Consider two vertices in either X or Y

$i, j \in X$ OR $i, j \in Y$

consider shortest v,i- and v,j-paths



Q: can edge (i, j) exist?

as P_j and P_i are both odd/even:

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consider a walk W:

- from v to i via p_i
- from i,j via hypothetical edge
- from j to v via p_j

What we want to show is that W odd therefor edge (i,j) can not exist

Note: parity %

consider length of W

odd + odd + 1 = odd

even + even + 1 = odd

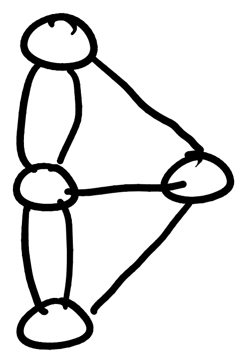
From our prior proof:

A closed odd walk contains an odd cycle

=> this implies edge (i,j) can not exist can not exist

=> as no such edge among vertices in X or vertices in Y can exist, our X, Y sets are valid bipartition □

Recall: Euler and his bridges

as a graph 

Euler: does a closed trail exist that traverses all edges

(Euler Tour / Circuit / Trail)

We want to "Characterize" an Euler Tour => define the properties of some G that contains an Euler Tour

The properties of some G
that contains an Euler Tour

First: show if $v \in V(G) : d(v) \geq 2$

$\Rightarrow \exists C_n \subseteq G$ as a subgraph

implies

some cycle of length n

To do this: we'll construct an **extremal argument** (using the extremal principle)

Extremal principle: with some set of countable/orderable values (finite) (well orderable) $\exists$ some item with a maximum value and some item with a minimum value

$S = \{1, 2, \underline{10}, 9, -1, \underline{-5}\}$

max min

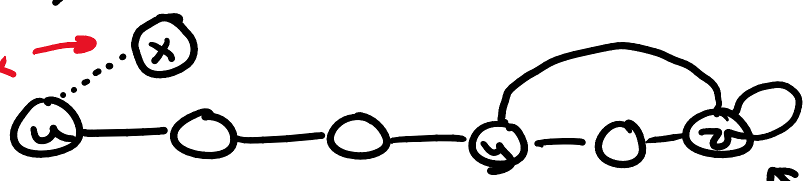
For our proof, select $P \subseteq G$,

where P is a path of maximum length in G

(consider P's structure)

Q: can x exist

x →



same with v

note: u has $d(u) \geq 2$

this is a contradiction against our selection of P

(there exists x,v-path which is longer than P)

Hence: both u, v have an edge to same $y \in P$

$\Rightarrow$ this creates a cycle $\square$

What are the necessary conditions of a graph with an Euler Tour?

1. G has at most 1 nontrivial connected component

(otherwise, how can a trail be connected)

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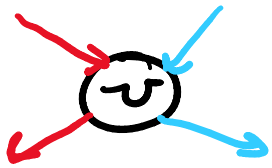
2. $\forall v \in V(G) : d(v) = \text{even}$



every time we reach a vertex on our trail, we exit via some other edge

Q: are these necessary conditions also sufficient?

$\boxed{G \text{ is Eulerian} \iff \forall v \in V(G) : d(v) \text{ is even and } G \text{ has 1 nontrivial component}}$