Lecture 5 - Degrees, Graphic Sequences, Digraphs

Tuesday, January 23, 2024 7:00 PM

Let's prove:

G has at most 1 nontrivial comp
and $\forall v \in V(G): d(v) = \text{even}$
$\Rightarrow$ G is Eulerian

We'll do strong induction on $|E|$

Basis $P(0)$: $0 \rightarrow$ trivial tour $\{\}$

Assume we have some $P(n) \in \mathbb{C}$

specified graph class

Note: minimum degree must be at least 2

$\rightarrow$ from last class: $\exists C_n \in G$

exists some cycle of length n

$P(k) = P(n) - C_n$

Note: deleting a cycle subtracts exactly 2 degrees from each vertex $u \in V(C_n)$

Note x2: $P(k)$ might be

Note x2: $P(k)$ might be disconnected

→ we can still use our I.H. on all of $P(k)$'s components

→ we have some Euler tour on each component of $P(k)$

Q: How do we use that to prove Eulerianness of $P(n)$?

PROOF BY ALGORITHM

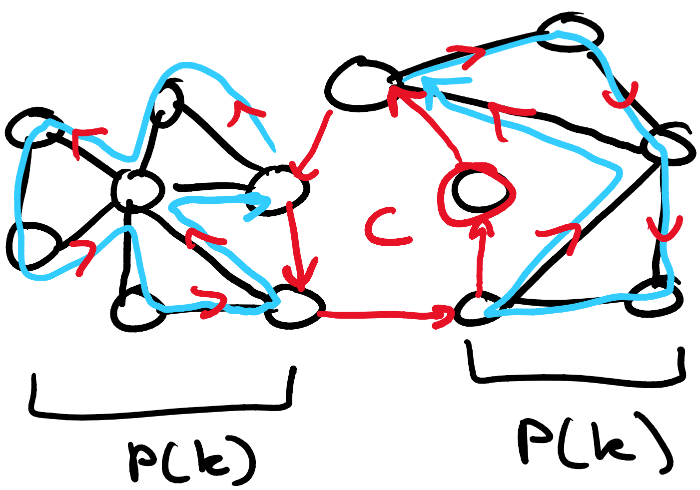
↪ we construct an algorithm that proves or guarantees some property

To complete our proof:
combine sub-tours on $P(k)$ with C_n to get a tour on $P(n)$

Our algorithm:
start at some $v \in C_n$
if $d(v) = 2$ traverse on C_n
else [illegible]

if $d(v) = 2$ traverse on C_n
else $\exists$ a tour from v into
a component of $P(h)$
→ follow that tour to completion
continue along C_n repeating
the above, but only traversing
edges on tours we have not
already taken

Output: Euler Tour on $P(n)$ □



Degrees

recall : $n = |V(G)|$
$m = |E(G)|$

degree of v → $d(v)$, d_v

for simple graphs: $d(v) = |N(v)|$

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For some G:

maximum degree $\rightarrow \Delta(G)$

minimum degree $\rightarrow \delta(G)$

G is k-regular if

$$\delta(G) = k = \Delta(G)$$

$$\forall v \in V(G): d(v) = k$$

Examples: all C_n are 2-regular

all K_n are $(n-1)$-regular

Degree sum formula:

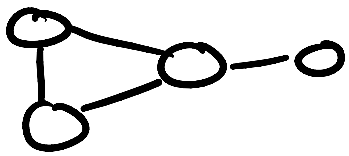
$$\sum_{v \in V(G)} d(v) = 2m = 2|E(G)|$$

why: each edge adds +1 degree to each endpoint

Q: what are the possible valid degrees for some G?

Degree Sequence: list of all

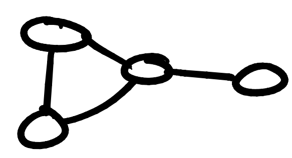
Degree Sequence: list of all degrees for vertices in some G

G  $S = \{1, 2, 2, 3\}$

Graphic Sequence: a list of degrees that can realize a simple undirected graph

Realize: construct a graph with a given degree sequence

$S = \{1, 3, 2, 2\}$

↳ realize 

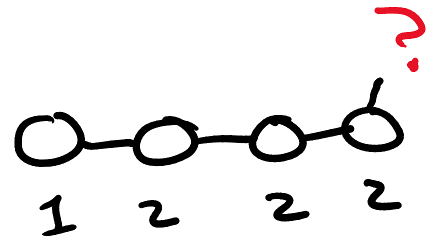
↳ S is graphic

Q: What sequences are graphic?

$S' = \{1, 2, 2, 2\}$

Note: $\Sigma S'$ is odd

Note: $\sum S'$ is odd

↳ 

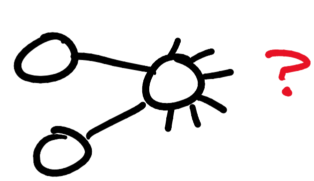
↳ can't realize S'

↳ S' is <u>not</u> graphic

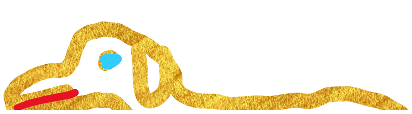
Takeaway: an even degree sum is a <u>necessary</u> condition for a graphic sequence

Q: is it <u>sufficient</u>?

Proof by **counter-example**

$S'' = \{8, 1, 1\}$ 

How can we tell if a sequence is graphic?



Havel-Hakimi

Big dog of GT

Havel-Hakimi Theorem (/algorithm)

→ Given a non-increasing sequence $S = \{d_1\ d_2 \dots d_n\}$

$$d_1 \geq d_2 \geq \dots \geq d_n$$

S is graphic iff

$$S' = \{(d_2 - 1)(d_3 - 1) \dots (d_{(d_1+1)} - 1) \dots (d_n)\}$$

is graphic

Example:

$$S = \{\cancel{3}, \underset{-1}{2}, \underset{-1}{2}, \underset{-1}{1}\}$$

$$S' = \{\cancel{1}, \underset{-1}{1}, 0\}$$

$$S'' = \{0, 0\}$$

Ways we can tell some S is

Ways we can tell some S is not graphic via this process:

1. We end up with only a single nonzero value
2. We end up with negative values
3. We have some d_1 that is larger than the number of other nonzeros in the sequence

$$S = \{\cancel{3}, \underset{-1}{2}, \underset{-1}{2}, \underset{-1}{1}, 1, 1\}$$

$$s' = \{1\ 1\ 0\ 1\ 1\}$$

$$s' = \{\cancel{1}\ \underset{-1}{1}\ 1\ 1\ 0\}$$

$$s'' = \{0\ 1\ 1\ 0\}$$

$$s'' = \{\cancel{1}\ \underset{-1}{1}\ 0\ 0\}$$

$$s''' = \{0\ 0\ 0\}$$

Havel-Hakimi Algorithm

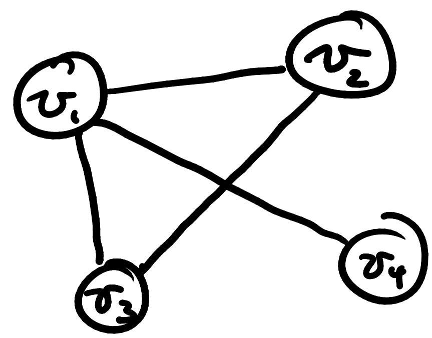
1. Map each value in S to some vertex

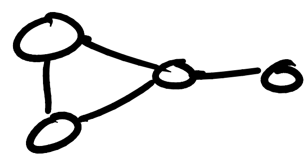
1. Map each value in S to some vertex
2. We draw an edge (d_i, d_j) when some d_j is decremented by the removal of d_i
3. Iterate until S^* is all zeros

$$S = \{ \overset{v_1}{\cancel{3}}, \underset{-1}{\overset{v_2}{2}}, \underset{-1}{\overset{v_3}{2}}, \underset{-1}{\overset{v_4}{1}} \}$$

$$S' = \{ \overset{v_2}{\cancel{1}}, \underset{-1}{\overset{v_3}{1}}, \overset{v_4}{0} \}$$

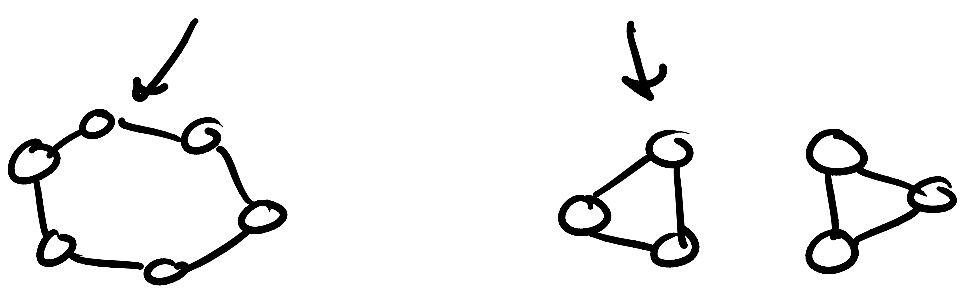
$$S'' = \{ 0 \; 0 \}$$



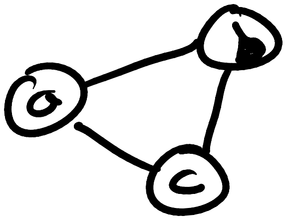


Note 1: a graphic sequence does not necessarily have a unique realization

$$S = \{2, 2, 2, 2, 2, 2\}$$



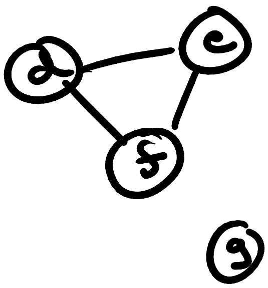
Note 2: it is necessary to sort



$$S = \{\overset{a}{\cancel{2}}\ \underset{-1}{\overset{b}{2}}\ \underset{-1}{\overset{c}{2}}\ \overset{d}{2}\ \overset{e}{2}\ \overset{f}{2}\ \overset{g}{2}\}$$

$$S' = \{\cancel{1}\ \underset{-1}{1}\ 2\ 2\ \overset{f}{2}\ \overset{g}{2}\}$$

swap order



$$S'' = \{\overset{d}{\cancel{2}}\ \underset{-1}{\overset{e}{2}}\ \underset{-1}{\overset{g}{2}}\ \overset{f}{2}\}$$

$$S''' = \{\overset{e}{\cancel{1}}\ \underset{-1}{\overset{f}{1}}\ \overset{g}{2}\}$$

$$S'''' = \{\overset{g}{2}\}$$

Braincercize: Can all possible realizations for a given graphic sequence be constructed via Havel-Hakimi?

via swapping order, we can construct multiple realizations

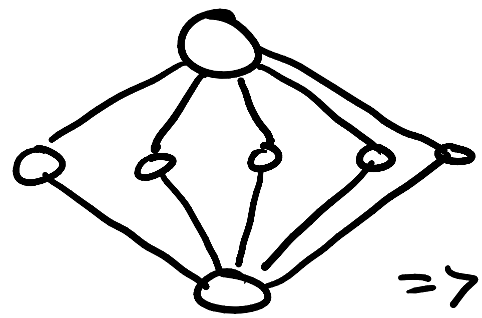
Proof by counter-example

a b c d e f g

$$S = \{\overset{a}{5}, \overset{b}{5}, \overset{c}{2}, \overset{d}{2}, \overset{e}{2}, \overset{f}{2}, \overset{g}{2}\}$$

via H-H, we will always connect the two largest degree vertices

However:



$\Rightarrow$ <u>NO</u>