5.1 Degrees

As mentioned previously, we're going to use variables n and m regularly as:

$$n = |V(G)|, m = |E(G)|$$

As we've discussed, the **degree** of a vertex is the number of times the vertex is the endpoint of an edge. We write degree of vertex v as $d(v)$ or sometimes d_v. For a graph G, the maximum degree is $\Delta(G)$ and the minimum degree is $\delta(G)$. A graph is **regular** if $\Delta(G) = \delta(G)$. A graph is k**-regular** if $k = \Delta(G) = \delta(G)$.

The degree sum formula shows that the sum of the degrees of all vertices in a graph is always even:

$$\sum_{v \in V(G)} d(v) = 2m$$

So it follows that there can only be an even number of vertices of odd degree in G. Our boi *parity* is making moves again.

The average degree of a graph G is $\frac{2m}{n}$. Therefore:

$$\delta(G) \leq \frac{2m}{n} \leq \Delta(G)$$

5.2 Graphic Sequences

The **degree sequence** of a graph is the list of vertex degrees, usually in non-increasing order: $d_1 \geq d_2 \geq \ldots \geq d_n$.

A **graphic sequence** is a list of nonnegative numbers that is the degree sequence of a simple graph. A simple graph G with degree sequence S *realizes* S. A degree sequence S is *realizable* if there exists some G with degree sequence S.

A sequence $S = \{d_1, d_2, \ldots, d_n\}$ is a graphic sequence iff sequence $S' = \{d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n\}$ is a graphic sequence, where $d_1 \geq d_2 \geq \ldots \geq d_n$ and $n \geq 2$ and $d_1 \geq 1$. This is called the **Havel-Hakimi Theorem**. We can use this general idea to also create (*realize*) a graph using a given graphic sequence.

For time consideration, we're not going to go over the proof in class, so go through the book or use other online resources to understand it. A couple relevant youtube videos are also listed below if you're interested:

https://www.youtube.com/watch?v=aNKO4ttWmcU
https://www.youtube.com/watch?v=iQJ1PFZ4gh0

5.3 Directed Graphs

Until today, we were only considering graphs with symmetric relations in the edges. Now, we're considering **directed graphs** or **digraphs**, where the edges have a defined directionality. The vertex where an edge starts is the **tail** and the vertex that is pointed to is the **head**. These together are the **endpoints**. We also term the tail as the **predecessor** of the head and the head as the **successor** of the tail. We can easily create a directed graph from an undirected graph by *orienting* each edge. An **orientation** of an undirected graph involves the selection of a direction for each edge, to create a directed graph.

As with our undirected graph classes, we can consider digraphs as **simple digraphs** if they don't have repeated edges or loops. Note that a simple digraph can have two edges between the same two vertices as long as they point in opposite directions. *Loopy digraphs* contain directed loops and *multi-digraphs* can contain multiple edges of the same directionality between the same two vertices.

We have similar definitions in directed graphs for **walks**, **paths**, **trails**, and **cycles**. Likewise, we have the same concepts of **subgraphs** and **isomorphism**. The **adjacency** matrix is created in a similar row-wise fashion, where a nonzero in position (x, y) indicates one or more edges pointing from vertex x to vertex y. Generally, the adjacency matrix is not guaranteed to be symmetric.

Instead of just one measure of degree degree, digraphs consider both **out degree** $(d^+(v))$ or **in degree** $(d^-(v))$. We also have the out neighborhood $(N^+(v))$ or successor set and the in neighborhood $(N^-(v))$ or predecessor set.

5.4 Directed Graph Degrees

For directed graphs, we've already seen that we consider both **out degree** $(d^+(v))$ or **in degree** $(d^-(v))$ separately. We likewise have minimum and maximum out and in degrees:

$$\delta^-(v), \delta^+(v), \Delta^+(v), \Delta^-(v)$$

And our degree sum formula for digraphs:

$$\sum_{v \in V(G)} d^+(v) = |E(G)| = \sum_{v \in V(G)} d^-(v)$$

As we treat the degrees of vertices in a digraph as pairs (out degree, in degree), we define the **degree sequence** for digraphs as a list of such pairs.

$$S = \{(d^+(v_1), d^-(v_1)), (d^+(v_2), d^-(v_2)), \ldots (d^+(v_n), d^-(v_n))\}$$

We have a similar notion of realizability, given the above. Let's prove that *a list of pairs of nonnegative integers is realizable as a degree sequence of a directed graph if and only if the sum of all first values in the pairs equal the sum of all second values in the pairs.* Note that here, we can consider multi-edges in our realization.

Even further, we can consider **Eulerian digraphs**. Similar to before, a digraph is Eulerian if there exists a closed directed trail containing all edges. As the proof for the directed case is identical to the undirected case, we leave it as an exercise for the reader (or as a question on a future quiz, homework, or exam).