Directed graphs
aka <u>digraphs</u>

We consider directionality for all edges in a digraph

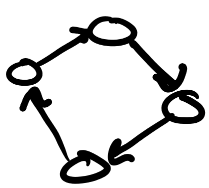
$u \longrightarrow v$ $\quad e = (u, v)$

tail — head

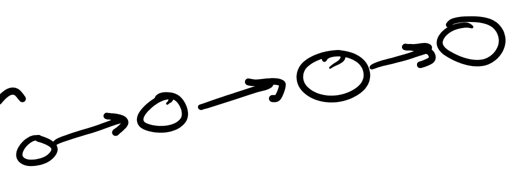
predecessor — successor

For walks, trails, paths, cycles
→ same definition as with undirected graphs, but now they follow the directions of the edges

directed cycle

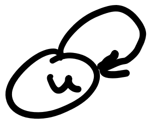


path



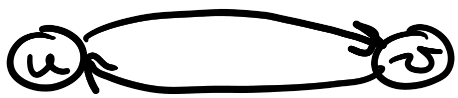
We also have the notion of simple, loopy, and multi digraphs

Note: a digraph is still simple if we have edges (u,v) and (v,u)



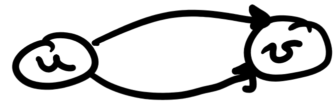
loop

(u,u)



simple

$(u,v),(v,u)$



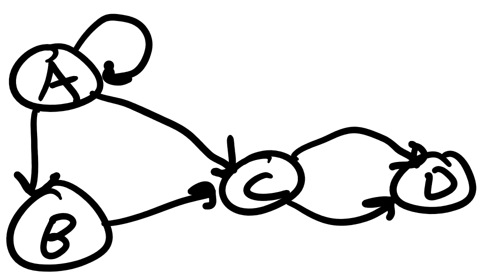
multi-edge

$(u,v),(u,v)$

Adjacency matrix

No longer symmetric

A nonzero at (i,j) implies the existence of an edge from i to j



$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Σrow_i = out degree of i

Σcol_i = in degree of i

out degree $d^+(v)$ = # of edges out of v

out degree $d^+(v) = \#$ of edges out of v

in degree $d^-(v) = \#$ of edge in to v

$N^+(v)$ = out neighbors of v

$N^-(v)$ = in neighbors of v

$\Delta^+(G), \Delta^-(G)$ = max out/in degree of graph G

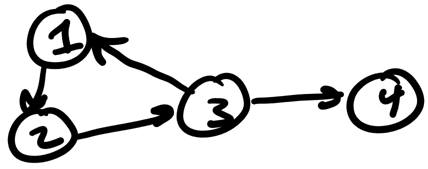
$\delta^+(G), \delta^-(G)$ = min out/in degree

Degree sum formula

$$\sum_{v \in V} d^+(v) = \sum_{v \in V} d^-(v) = m = |E|$$

Degree sequences

We now consider out and in degree pairs (out, in)



$S = \{(1,1), (1,1), (2,1), (0,1)\}$

Q: How can we tell if some degree sequence is graphic?

Necessory condition:

sum of the in degrees equal to the sum of the out degrees

Q: Is this condition sufficient?

For simple digraphs

→ No

For loopy multigraphs

→ Yes

PROOF BY ALGO.

consider $(d_i^+, d_i^-) : 1 \le i \le n$

$$m = \sum d_i^+ = \sum d_i^- : 1 \le i \le n$$

consider m lines

d_i^+ dots get labeled i

d_i^- dots get labeled $-i$

Consider n vertices $1 \ldots n$

$v_1, \ldots v_n$

For each line

construct an edge as a positive

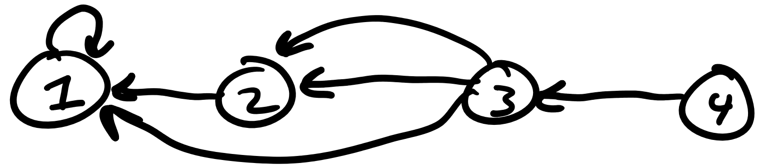
construct an edge as a positive label to a negative label corresponding to the vertex set

$$S = \{ \overset{i=1}{(1,3)}, \overset{i=2}{(1,2)}, \overset{i=3}{(3,1)}, \overset{i=4}{(1,0)} \}$$

1	2	3	3	3	4	outs
↓	↓	↓	↓	↓	↓	
1	1	1	2	2	3	ins

$$E = \{ (1,1), (2,1), (3,1), (3,2), (3,2), (4,3) \}$$

vertex labels



Q: can we produce all possible realizations of a given degree sequence?

→ Yes, by permuting the head and tail labels in our edge list

Eulerian Digraphs

A digraph is Eulerian iff there has a directed trail containing all edges

A digraph is Eulerian iff there is at most one nontrivial component and $\forall v \in V: d^-(v) = d^+(v)$

Proof: same proof as with the undirected case

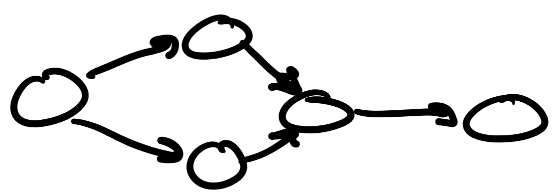
Trees 

Tree: a connected undirected simple acyclic graph

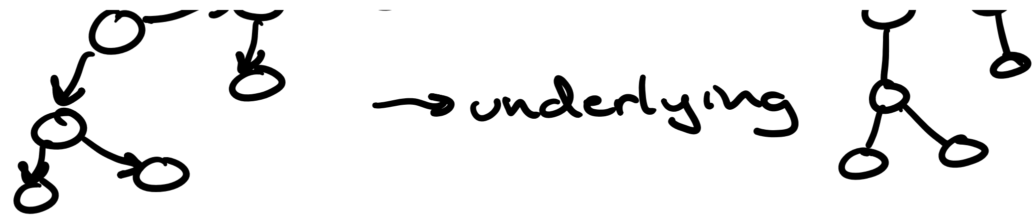
↳ contains no cycles

Forest: an undirected simple acyclic graph

DAG: directed acyclic graph



Polytree: a DAG where the underlying graph is a tree



Tree T necessary conditions

T is <u>minimally</u> connected

↳ $T-e$ is disconnected for all $e \in E(T)$

T is <u>maximally</u> acyclic

↳ $T+e$ create a cycle $e=(u,v)$ $\forall u,v \in V(T)$

T has $|E(T)| = |V(T)| - 1$

T has a single unique u,v-path $\forall u,v \in V(T)$

T is bipartite

Braincercize: which of the above properties are also sufficient?

Weak induction on trees

T is a tree $\rightarrow$ T is bipartite

Weak induction on trees

Prove: T is a tree → T is bipartite

weak induction on $|E(T)|$

or $|V(T)|$

Basis $P(0) \to 0$ ✓

Assume we have $P(k)$ as a tree

and via I.H. $P(k)$ is bipartite

Add an edge (and vertex) to

create $P(k+1)$

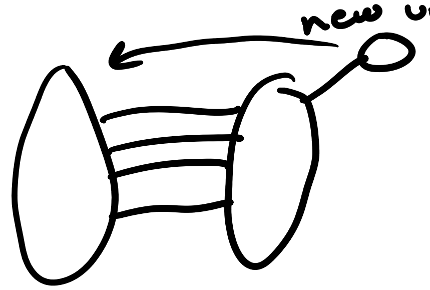
Note: we have a bipartitioning

of $P(k)$

Note 2: $P(k+1)$ adds a new

leaf to $p(k)$

new vert → leaf: a degree-1 vertex



X Y

→ we can place the

new leaf into the

opposite partite set

of its neighbor

$\Rightarrow P(k+1)$ is bipartite □

Note: we can equivalenty show the

above via strong induction

above via strong induction
using deletion of a leaf as
our construction

NOW: strong induction

Prove: T is a tree $\rightarrow \forall u,v \in V(T)$:

$\exists$ a unique u,v-path

strong induction on $|V(T)|$

Basis P(1): o ✓

Consider tree P(n)

Construct $P(k) = P(n) - \ell$, where
ℓ is a leaf vertex

Our construction to produce P(k)
will not introduce a cycle and
will not disconnect the tree

→ I.H. on P(k) gives us unique
u,v-paths $\forall u,v \in V(T)$

Bring it on back to P(n)

Bring it on back to $P(n)$

→ by adding back our deleted leaf

Bring it on home

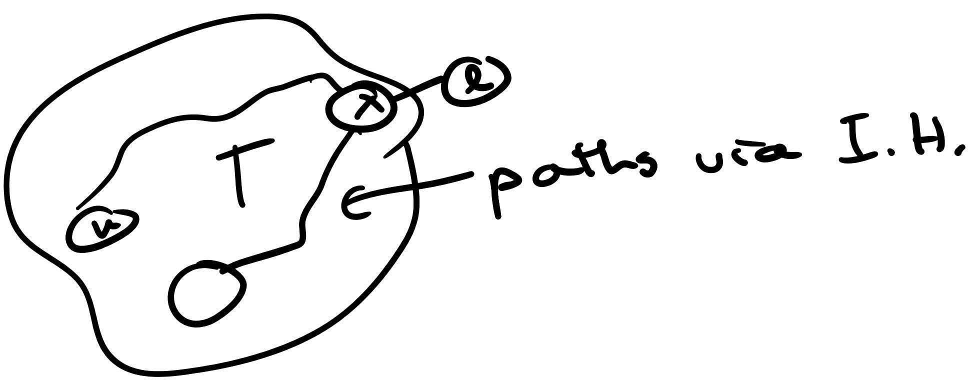
→ by showing our result via I.H. still holds on $P(n)$

We already know

$\{\forall u,v \in V(P(n)) : u,v \neq \ell\} : \exists$ unique u,v-path

To get the rest of the ℓ,v-paths

→ we can add edge e to the unique u,x-paths, where $x \in N(\ell)$ and v is all other vertices in $V(T)$ $\square$



Extra fun definitions

distance: $d(u,v)$ = length of the

distance: $d(u,v)$ = length of the shortest u,v-path

diameter: $D(G)$ = length of the longest u,v-path $\forall u,v \in V(G)$

lil' epsilon

eccentricity: $\varepsilon(v) = \max\limits_{\forall u \in V(G)} d(u,v)$

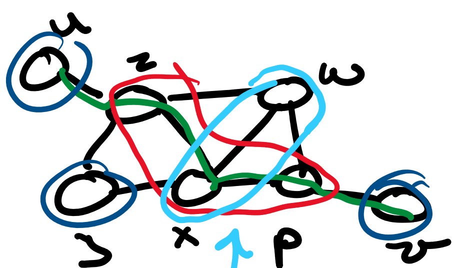
radius: $R(G) = \min\limits_{\forall u \in V(G)} \varepsilon(u)$

Center of G = the induced subgraph of vertices with the minimum eccentricity of G

$\underbrace{\{\forall u \in V(G) : \varepsilon(u) = R(G)\}}$

↳ center is subgraph induced on this defined vertex set

G



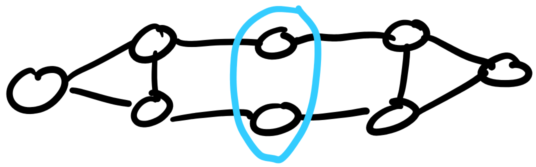
center of G = {x, w} as induced subgraph

$d(u,v) = 4$

$D(G) = 4$

$\varepsilon(w) = 2$

$R(G) = 2$



Note; the center is not necessarily connected