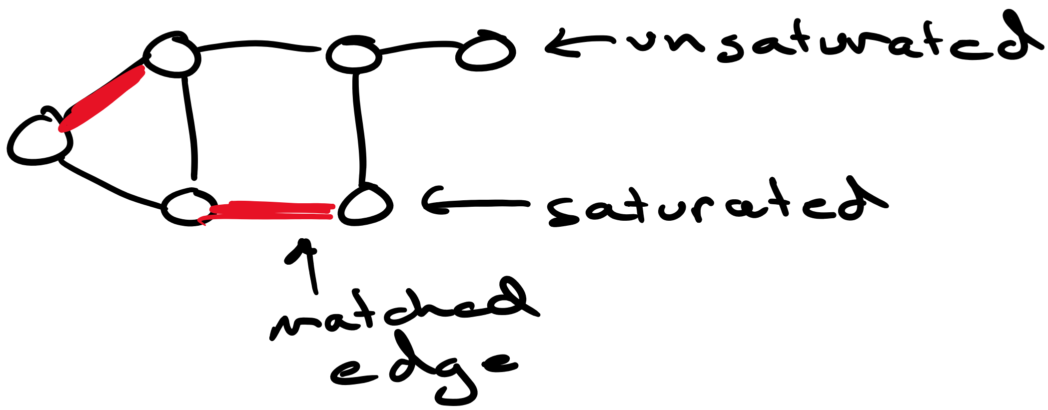
Lecture 11 - General Graph Matching

Thursday, February 15, 2024 9:46 AM

Review

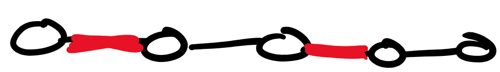
Match: set of edge on same G
with no shared endpoints



maximum: largest possible match
maximal: can't be made larger
perfect: saturate all vertices

Berge: M is maximum on G
iff G has no M-aug paths

M-aug: 

M-alt: 

Hall: $\exists M$ that saturates X
in an X, Y-bigraph $|X| \leq |Y|$

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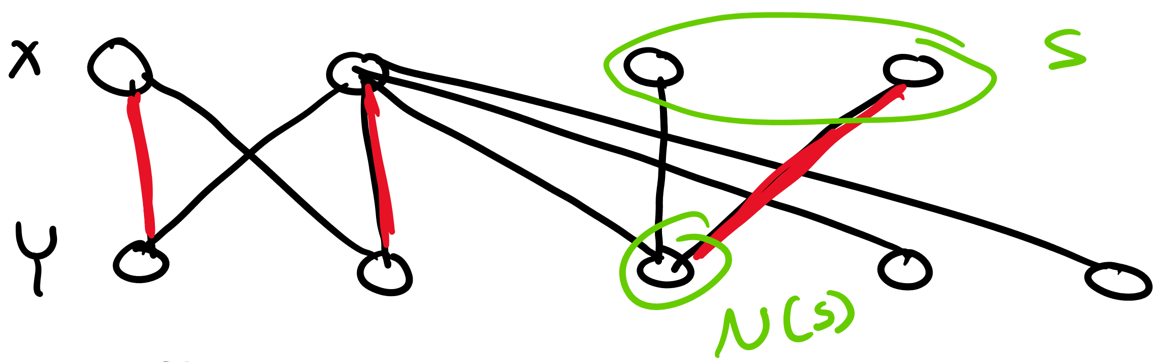
iff $\forall S \subseteq X : |N(S)| \ge |S|$



Q: Is this match optimal?

(is it maximum?)

A: Yes

contrapositive of Hall

$P \Rightarrow Q$

$\neg Q \Rightarrow \neg P$

No X-saturating match

iff $\exists S \subseteq X : |S| > |N(S)|$

Also: maximum weight matching

→ maximize sum of edge weights in a match

stable matching

→ given preferences, no

→ given preferences, no unstable pair (x,a)

x and a have other matched partners, but both prefer the (x,a) match

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General Graph Matching

$o(G)$ = # of odd components of G

odd # of vertices

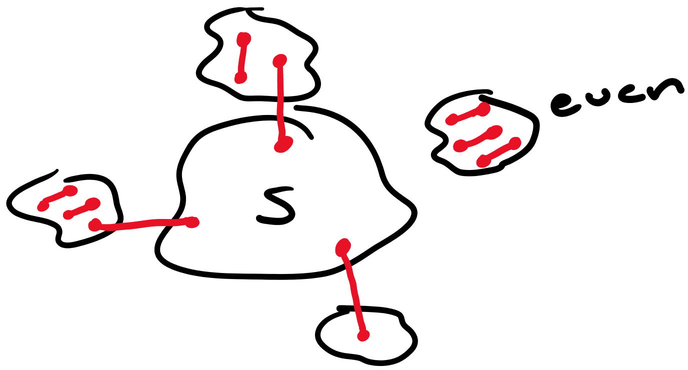
Tutte: G has a perfect match iff $\forall S \subseteq V(G): o(G-S) \leq |S|$

G has P.M. $\Rightarrow \forall S \subseteq V(G): o(G-S) \leq |S|$

- consider some S and $G-S$

Note: each odd component of $G-S$ cannot be perfectly matched

→ at least one vertex in each component must be matched component must be matched to some vertex in S



$\Rightarrow$ so $|S|$ must be bound below by $o(G-S)$ ✓

$\forall S \subseteq V(G): o(G-S) \leq |S| \Rightarrow G$ has P.M.

Contrapositive

G has no P.M. $\Rightarrow \exists S \subseteq V(S): |S| < o(G-S)$

Note: condition holds if we add edges to G

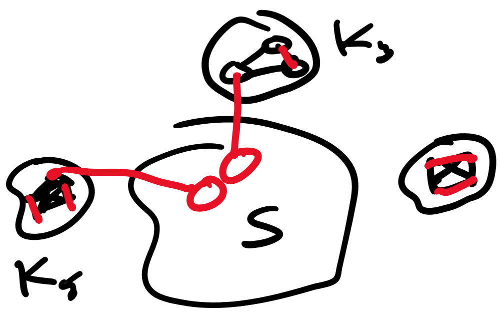
Extremal

we will consider an extremal choice of $G \to G'$ where G' is edge-maximal

$G \to G'$ where G' is edge-maximal with respect to having no P.M.

Define $S = \{\forall v \in V(G') : d(v) = |V(G')| - 1\}$

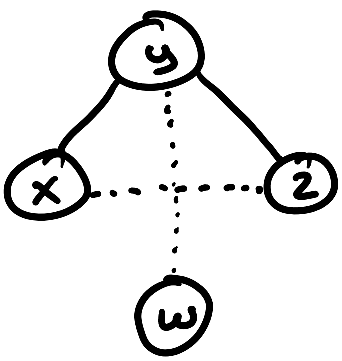
Case 1: $G' - S \to$ all components are cliques



Note: S must be "bad", as we could otherwise construct a P.M.

$|S| < o(G-S)$

Case 2: $G' - S \to$ not all cliques



$\exists x, z$ s.t. $(x,z) \notin E(G'-S)$

$\exists y$ s.t. $(x,y)(z,y) \in E(G'-S)$

$\exists w$ s.t. $(y,w) \notin E(G'-S)$

From our selection of G'

$G' + (y,w)$ creates a P.M.

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$G' + (x, z)$ creates a P.M.

We'll show that this implies a P.M. on G itself

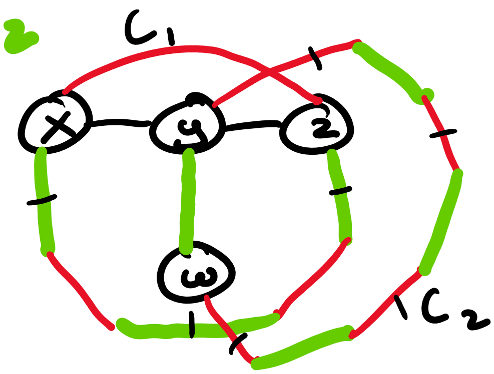
Define:

M_1 = P.M. on $G' + (x, z)$
M_2 = P.M. on $G' + (y, w)$

$F = M_1 \Delta M_2$ (XOR) $\rightarrow$ must be paths or cycles

However: as M_1 and M_2 are P.M.s, we only have cycles

M_1
M_2



C_1 = cycle with (x, z)

C_2 = cycle with (y, w)

Case 2a: $C_1 \neq C_2$

P.M. on G' = all $e \in M_2, e \in C_1$

P.M. on G = all $e \in M_2$, $e \in C_1$
all other M_1

→ P.M. without using (y,w) or (x,z)

contradiction

$\Rightarrow$ S must be "bad"

Case 2b: $C_1 = C_2$

P.M. on G' = M_1 on C_2 from w until x or z

if we reach x:
P.M. on G' += (x,y) + M_2 from y to z

if we reach z:
P.M. on G' += (y,z) + M_2 from y to x

→ either way we have a P.M. w/o (x,z) or (y,w)

contradiction

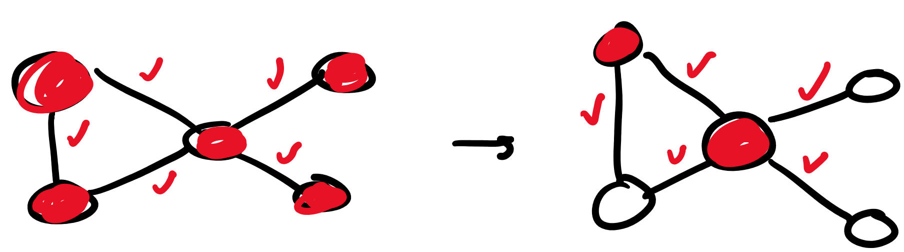
Contradiction

$\Rightarrow$ S must be "bad" for all cases $\square$

$\boxed{\text{Tutte: } G \text{ has a P.M. iff } \forall S \subseteq V(G): o(G-S) \leq |S|}$

Vertex and Edge Covers

vertex cover: a set $Q \subseteq V(G)$ that has at least one end point $\forall e \in E(G)$

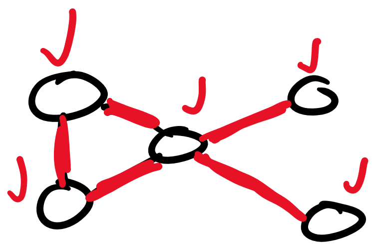


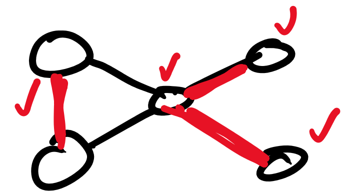
V(G) trivially a vertex cover

minimum cover

edge cover: a set $C \subseteq E(G)$ that has at least one edge incident on all $v \in V(G)$

edge incident on all $v \in V(G)$





E(G) is trivially an edge cover

minimum edge cover

Note: lower bound on $|C|$ is $|C| \geq \left\lceil \frac{|V(G)|}{2} \right\rceil$

König-Egerváry: on a bipatite graph, the size of a minimum vertex cover equals the size of a maximum match

$|M_{max}|$ = max match

$|Q_{min}|$ = min cover

K.E. $|M_{max}| = |Q_{min}|$ on bigraph G

Note: $|Q| \geq |M|$ for any cover and match

→ every matched edge needs to be covered by at least one $v \in Q$

→ This is a good example of a min-max relation aka dual optimization problem

↳ Solution to the minimization problem is the upper bound to the maximization problem and vice-versa (sp?)

Dominating sets

<u>Dominating set</u>: $D \subseteq V(G)$ such that $\forall v \in V(G) : v \notin D$
$\rightarrow \exists u \in N(v) : u \in D$

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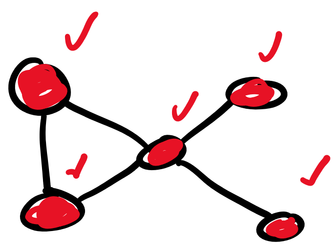
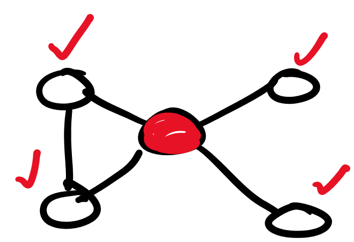
aka

every vertex is either in D or has a neighbor in D



trivially V(G) is a D.S.



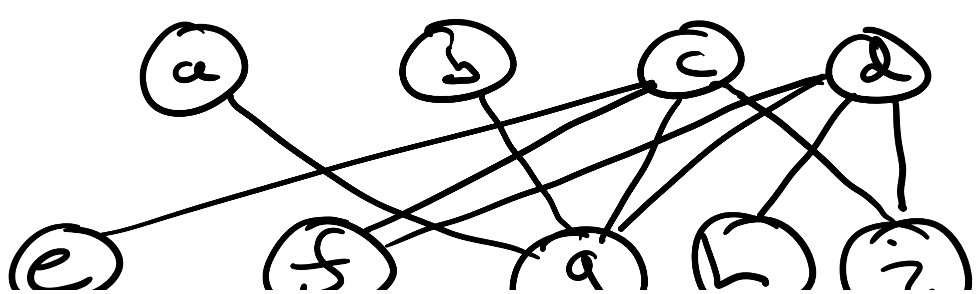
minimum D.S.

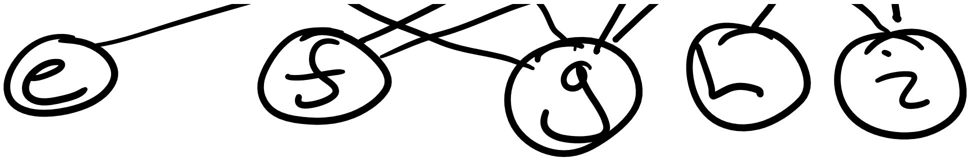
Weekly Problem 6:

P1: find and prove maximum matches on bipartite graphs

P2: Prove that in the stable matching algorithm no man gets rejected by all women.

P1a)





P2 b)