Lecture 12 - Connectivity

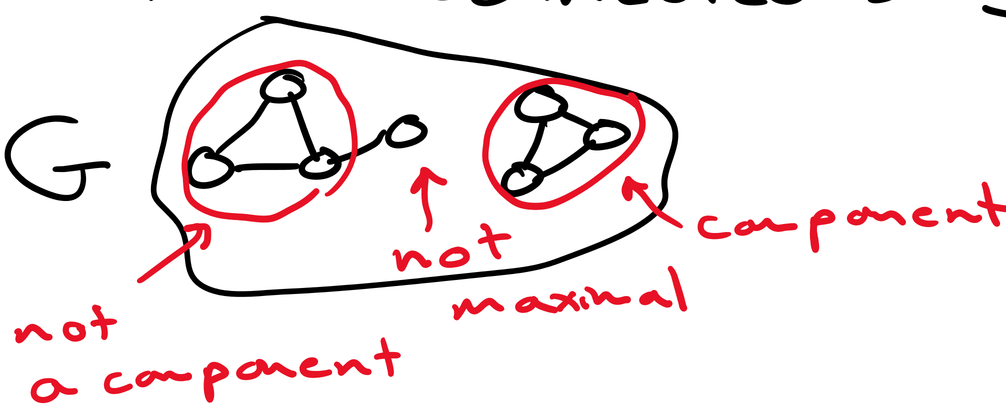
Friday, February 16, 2024 7:02 PM

Review of connectivity

G is <u>connected</u> if

$\forall u,v \in V(G): \exists\ u,v$-path

<u>connected component</u> is a maximal connected subgraph



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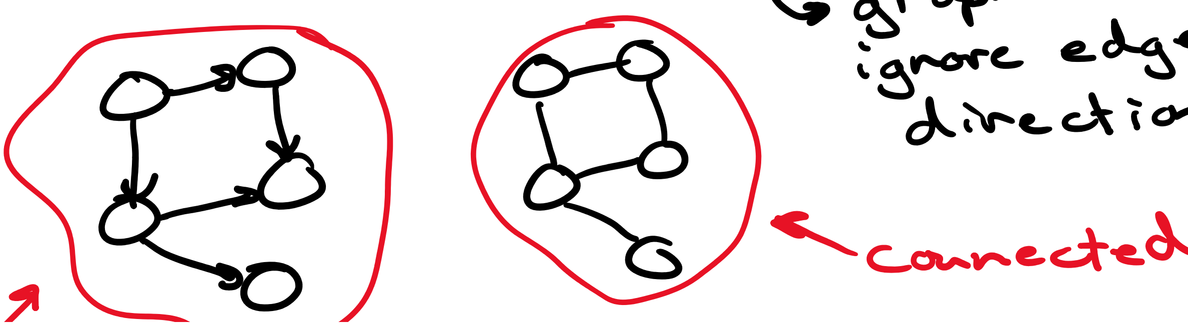
weak connectivity

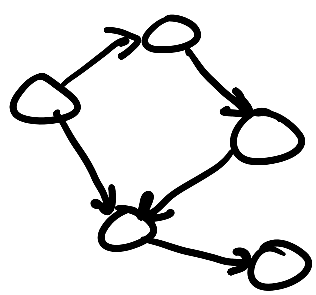
A digraph is <u>weakly connected</u> if the <u>underlying graph</u> is connected

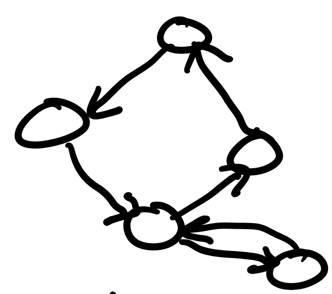
graph if we ignore edge directions



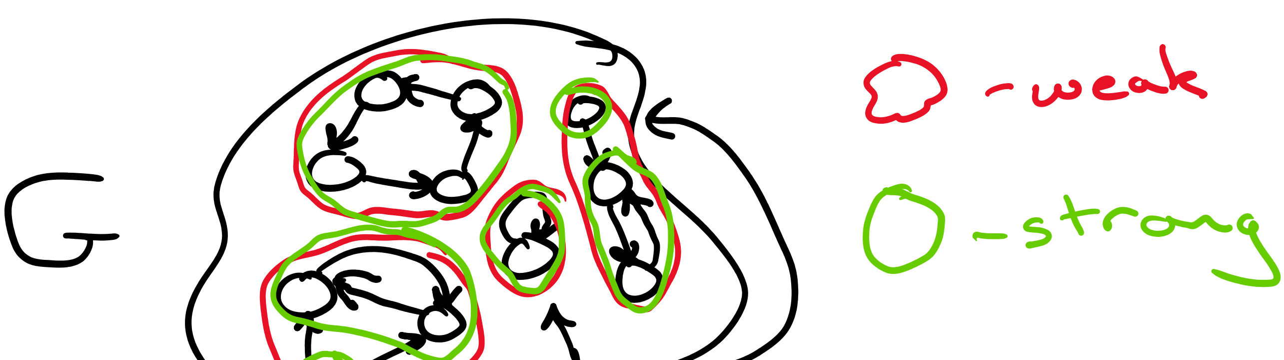
Strong Connectivity

A digraph is <u>strongly connected</u> if $u, v \in V(G)$: $\exists\, u,v$-path (directed path)





<u>weak / strong components</u> - a maximal weakly / strongly connected subgraph



trivial <u>component</u>: a

trivial component: a component of order 1

Vertex Connectivity
(undirected graphs)

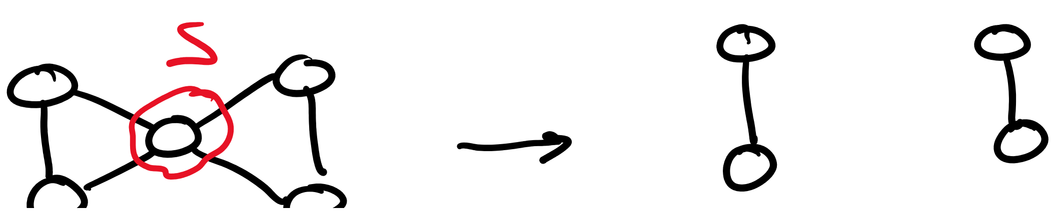
Recall: cut vertex is a vertex in G s.t. $G - v$ has more components than G

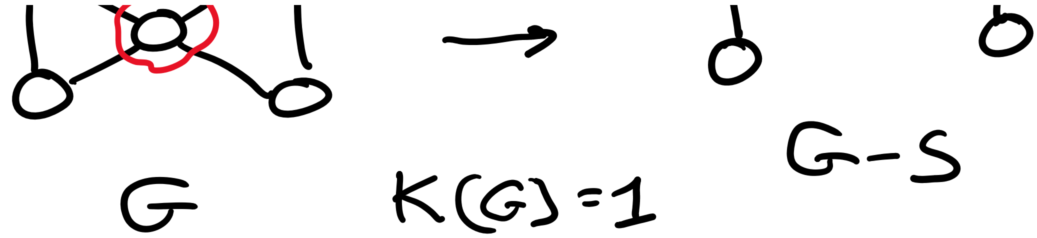
Separating (sp?) set: a set $S \subseteq V(G)$ s.t. $G - S$ has more components than G

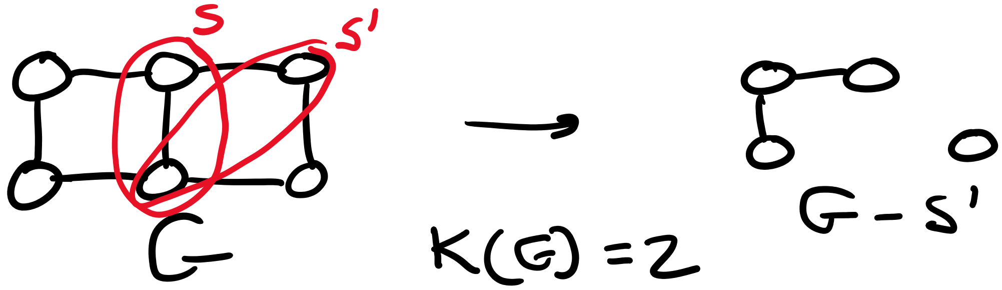
AKA: vertex cut
vertex separator

Connectivity of $G = \kappa(G) = k$ is the size of a minimum vertex cut

G is k-connected if $\kappa(G) = k$







Note: for connectivity, the maximum size of a separator is $|V(G)|-1$

$\rightarrow$ K_n is $(n-1)$-connected

Also: C_n is 2-connected
$(n \geq 3)$

Tree T is 1-connected

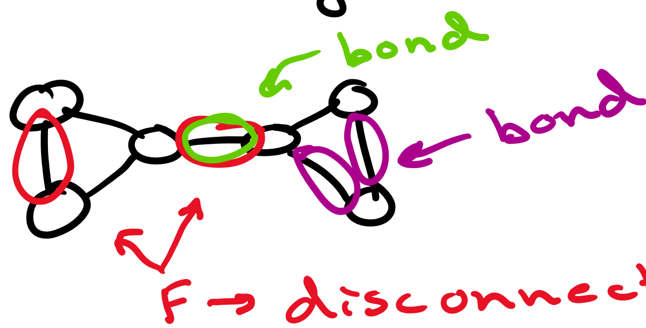
Edge Connectivity

<u>cut edge</u>: an edge in some G s.t. $G-e$ has more components than G

Disconnecting set: a set of edges $F \subseteq E(G)$ s.t. $G - F$ has more components than G

AKA: edge cut*
edge separator

Bond: minimal edge cut



edge connectivity of $G = \kappa'(G) = k$
minimum size of an edge cut
→ G is k-edge-connected

K_n is $(n-1)$-edge-connected
T is 1-edge-connected
C_n is 2-edge-connected
$(n \geq 2)$

Note: if G is k-connected
then G is also $(k-1)$-connected

Note: if G is k-connected
then G is also (k-1)-connected
(k-2)-connected... 1-connected

→ same with edge-connectivity

Bounds on connectivity

$$\kappa(G) \overset{?}{\le} \kappa'(G) \overset{?}{\le} \delta(G)$$

Q: Can we place bounds on these values relative to each other?

→ Trivially, if we remove all edges incident on a minimum degree vertex, we will disconnect the graph

$$\kappa'(G) \le \delta(G)$$

Likewise, if we remove all neighbors of that minimum degree vertex, we disconnect G

$$\kappa(G) \le \delta(G)$$

But how are $K(G)$ and $K'(G)$ bounded relative to each other?

consider a minimum edge cut F that separates $V(G)$ into $S, \bar{S}$ s.t. $\bar{S} = V(G) - S$

Case 1: $\forall u \in S, \forall v \in \bar{S} : \exists (u,v) \in E(G)$

this implies

$$K'(G) = |F| = |S||\bar{S}| \geq |V(G)| - 1 \geq K(G)$$

so $K'(G) \geq K(G)$

Case 2: $\exists x \in S, \exists y \in \bar{S} : (x,y) \notin E(G)$

define $T = \{$ all $u \in N(x) : u \in \bar{S}$, and all $v \in S - x : \exists (v,z) \in E(G)$, $z \in \bar{S} \}$

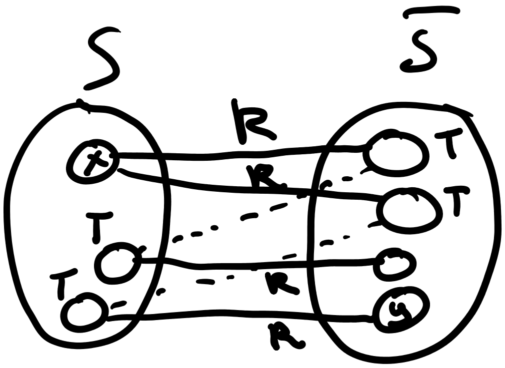
Note: all x,y-paths must go through some $w \in T$

(aka an x,y-separator)

define $R = \{$ all $e \in (x,w) : w \in T \cap \bar{S}$, and $f = (a,b) : a \in T \cap S, b \in \bar{S}$

(and we only select one

$f = (a,b) : a \in T \cap S,\ b \in \bar{S}$ (and we only select one edge per each a) }



Note: $|R| = |T|$

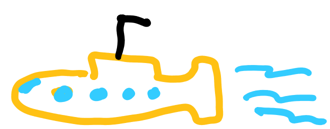
As we've only selected one f per each R out of multiple possible

$$|R| \le |F|$$

$$|T| \le |R| \le |F|$$

$$\kappa(G) \le \kappa'(G)$$

All together now



$$\boxed{\kappa(G) \le \kappa'(G) \le \delta(G)}$$