

## 12.1 Strong Connectivity of Directed Graphs

Earlier, we introduced the concepts of connectivity for undirected graphs and weak connectivity for directed graphs. An additional connectivity concept for digraphs is that of **strong connectivity**. The definition of strong connectivity is similar to regular connectivity in undirected graphs: for any  $u, v$  in a strongly connected component, there exists a directed  $u, v$ -path from  $u$  to  $v$ . Recall that weak connectivity of a directed graph is equivalent to connectivity of its underlying graph, where the **underlying graph** of a digraph is the undirected representation created by removing directionality from the directed edges.

## 12.2 Vertex Connectivity of Undirected Graphs

We're going to now somewhat generalize the concept of connectedness for undirected graphs in terms of network robustness. Essentially, given a graph, we may want to answer the question of how many vertices or edges must be removed in order to disconnect the graph; i.e., break the graph into multiple components.

Formally, for a connected graph  $G$ , a set of vertices  $S \subseteq V(G)$  is a **separating set** if subgraph  $G - S$  has more than one component or is only a single vertex. The set  $S$  is also called a **vertex separator** or a **vertex cut**. The **connectivity** of  $G$ ,  $\kappa(G)$ , is the minimum size of any  $S \subseteq V(G)$  such that  $G - S$  is disconnected or has a single vertex; such an  $S$  would be called a **minimum separator**. We say that  $G$  is  **$k$ -connected** if  $\kappa(G) \geq k$ .

## 12.3 Edge Connectivity

We have similar concepts for edges. For a connected graph  $G$ , a set of edges  $F \subseteq E(G)$  is a **disconnecting set** if  $G - F$  has more than one component. If  $G - F$  has two components,  $F$  is also called an **edge cut**. The **edge-connectivity** of  $G$ ,  $\kappa'(G)$ , is the minimum size of any  $F \subseteq E(G)$  such that  $G - F$  is disconnected; such an  $F$  would be called a **minimum cut**. A **bond** is a *minimal* non-empty edge cut; note that a bond is not necessarily a minimum cut. We say that  $G$  is  **$k$ -edge-connected** if  $\kappa'(G) \geq k$ . In a couple classes, we'll talk about how one might find a minimum cut in an arbitrary graph.

For a simple graph, we can show that  $\kappa(G) \leq \kappa'(G) \leq \delta(G)$ .