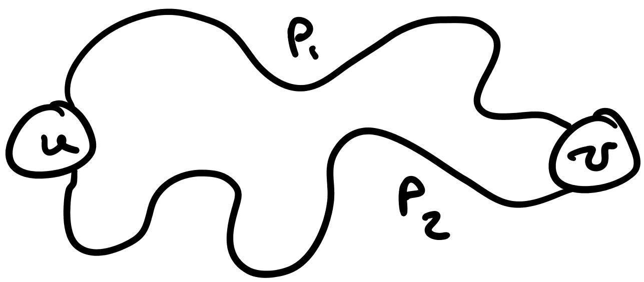
2-connectivity

→ must remove 2 vertices to disconnect G

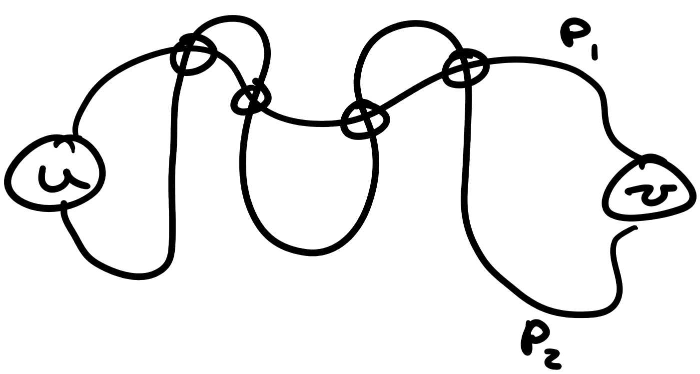
Internally disjoint paths

Paths between some u,v that share no internal vertices



Internally edge-disjoint paths

Paths between some u,v that share no internal edges



Whitney's Theorem

Whitney's Theorem

G where $|V(G)| \geq 3$ is at least 2-connected ~~iff~~

$\forall u, v \in V(G) : \exists P_1 P_2$ u,v-idps

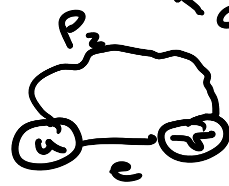
internally disjoint paths

($\Rightarrow$)

Induction on $d(u,v)$

distance

Basis: $P(1) \rightarrow$



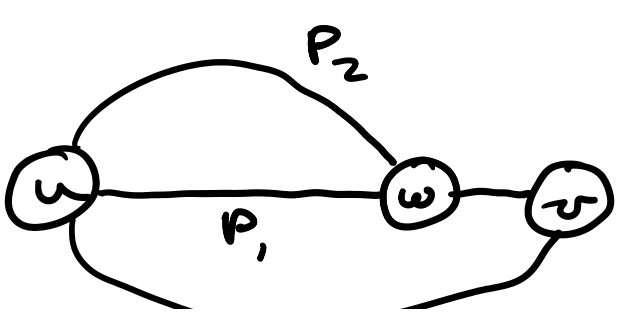
$P_1 = \{e\}$

$K(G) \leq K'(G)$ P_2 = any path on $G-e$

Consider $G = P(n)$ with $d(u,v) = n$

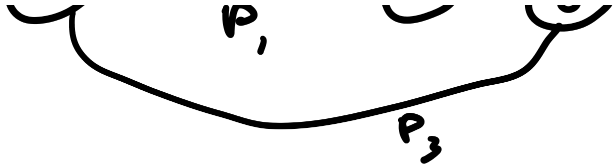
$\rightarrow \exists$ at least one u,v-path

on this path, consider $w \in N(v)$



$d(u,w) = n-1 = k$

I.H. $\rightarrow \exists P_1 P_2$-idps



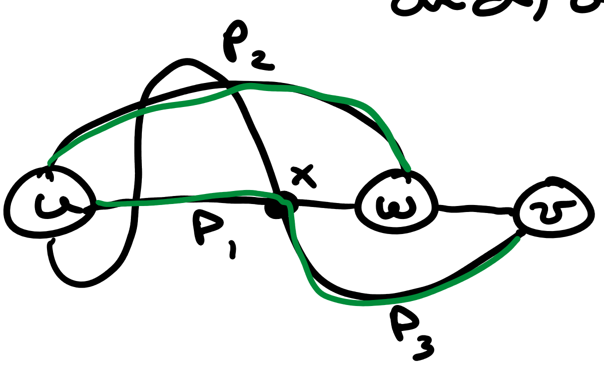
I.H. $\rightarrow \exists P_1 P_2$ - idps between u, w

Note: As G is 2-connected $\rightarrow \exists P_3 = u,v$-path on $G - w$

Q: Is P_3 internally disjoint to P_1 or P_2?

Case 1: Yes $\rightarrow P_3$ and $P_1 + (w, v)$ gives us 2 idps

Case 2: NO :($\rightarrow P_3$ intersects P_1 and/or P_2 some # of times

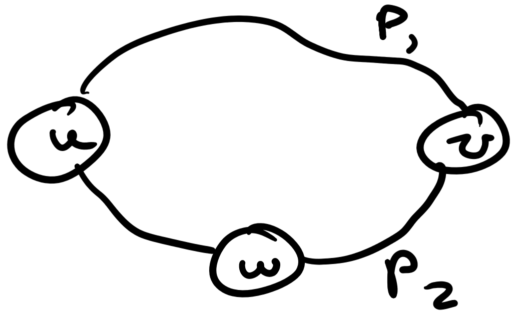


Define: x as the last vertex on P_1 or P_2 that P_3 intersects (wlog say its on P_1)

$\Rightarrow$ First path $= P_2$

Second path $= P_1$ until x and then P_3 until v

($\Leftarrow$) Consider any $u, v \in V(G)$ and their 2 u,v-idps



Consider any w on P_1 or P_2 and $G - w$

$\rightarrow$ there still at least one u,v-path

$\Rightarrow$ minimum vertex separator must be at least 2 vertices $\square$

2-connectivity equivalences

G is 2-connected $\Leftrightarrow \exists P_1, P_2$ idps $\forall u,v \in V(G)$

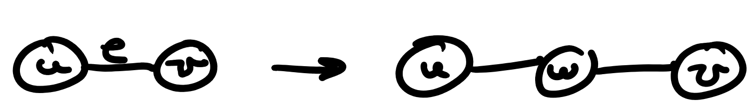
$\Leftrightarrow G$ has no cut vertex

$\Leftrightarrow \forall u,v \in V(G) : \exists C$ where $u,v \in C$

$\Leftrightarrow \forall e,f \in E(G) : \exists C$ where $e,f \in C$

To show the last statement, we'll use the concept of subdivisions

<u>subdivision</u>:



Note: subdivision preserves

Note: subdivision preserves 2-connectedness of G

So subdividing e, f to w, w' preserves our 2-connectedness and therefore 2 idps from w to w'

$\Rightarrow \exists$ s.t. $w, w' \in C$

and equivalently that cycle would have e, f in original G $\square$

Ear Decompositions

Recall: Decomposition of G is a listing of subgraph of G s.t. these subgraphs contain all $e \in E(G)$ exactly once

Open-ear decomposition

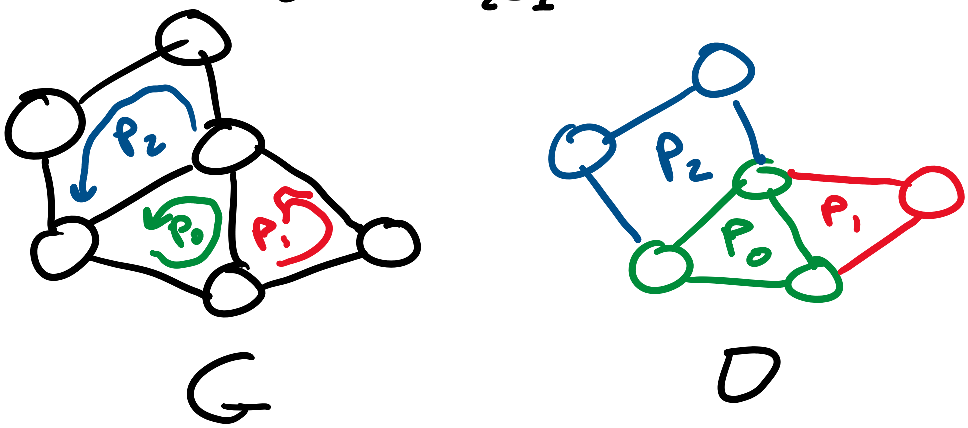
A decomposition of some G s.t.

$D = P_0\, P_1 \dots P_k$ ← subgraphs in the decomposition

$D = P_0 P_1 \ldots P_k$ in the decomposition

P_0 = cycle

P_i = a path whose endpoints are unique and exist on $P_0 P_1 \ldots P_{i-1}$



Q: Is the existence of an open ear decomposition necessary & sufficient for 2-connectedness of some G?

Prove: G is 2-connected $\Leftrightarrow$ G has open ear decomposition

($\Rightarrow$) PROOF BY ALGORITM

Select any $u, v \in V(G)$

$\hookrightarrow \exists C$ s.t. $u, v \in C$

$P_0 = C = D$

$\hookrightarrow \exists C$ s.t. $u, v \in C$

$P_0 = C = D$

while $\exists e \in E(G) : e \notin P_0 \dots P_{i-1}$

consider any $f \in P_0 \dots P_{i-1}$

$\hookrightarrow \exists C'$ s.t. $e, f \in C'$

P_i = e follows C' is both directions until some vertices in $P_0 \dots P_{i-1}$ are reached

$D \leftarrow P_i$

=> this builds our decomposition ✓

(<=) To show: prove adding an ear to an existing open ear decomp. retains 2-connectness of that decomposition

First consider P_0

$\hookrightarrow$ cycles are 2-connected

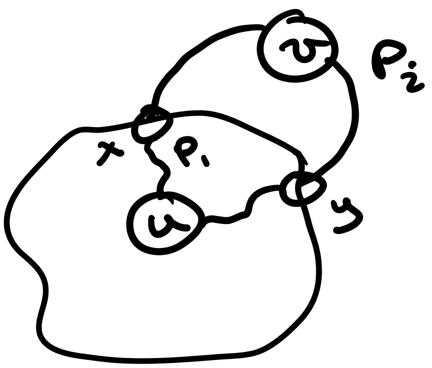
Next, consider some P_i open ear

and some $v \in P_i$

and some $u \in P_0 \dots P_{i-1}$

and some $u \in P_0 \ldots P_{i-1}$

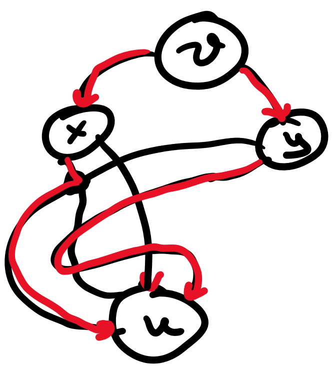
Q: can we find 2 u,v-idps?



$P_0 P_1 \ldots P_{i-1}$

P_1 = v to x, then x to u along one u,x-idps

P_2 = v to y, then y to u along some u,y-idps



Q: can we guarantee that both P_1, P_2 are internally disjoint?

A: Yes, using same basic logic from our first proof

=> Adding an ear to a 2-connected graph will retain 2-connectedness so a decomposition imply 2-connectedness of G $\square$

2-edge-connectivity

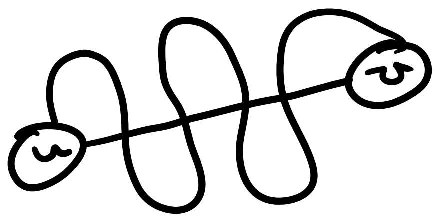
G is 2-edge-connected if the size of a minimum cut is 2 edges

↳ similar ideas to 2-connectivity, in that we can guarantee the existence of a closed trail containing any pair of vertices (or edges)

G is 2-edge-connected $\iff$

$\forall u, v \in V(G) : \exists\, u,v$-edps

internally edge disjoint paths



G is 2-edge-connected $\iff$

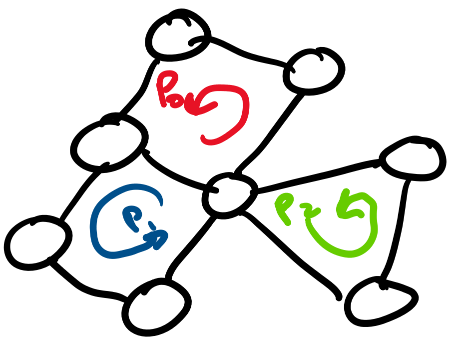
G has a closed ear decomposition

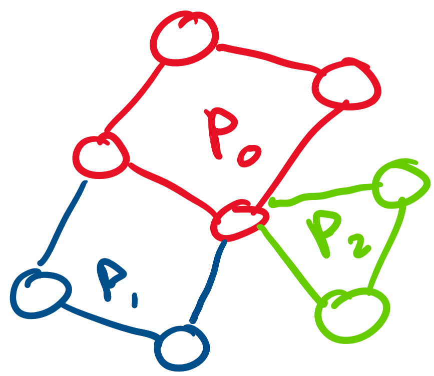
Closed ear decomposition

P_0 = cycle

P_i = open or closed path

P_i = open or closed path whose endpoints are on $P_0 P_1 \dots P_{i-1}$





closed ear, starts and ends at the same $v \in P_0 \dots P_{i-1}$

G is 2-edge-connected
$\iff$ G has no cut edge
$\iff \forall u, v \in V(G) : \exists$ 2 u,v-edps
$\iff$ G has a closed ear decomposition

Biconnectivity

→ a biconnected graph G where $|V(G)| \geq 3$ is 2-connected

OR K_1 or K_2 is G

Block Decomposition of G

blocks are maximal biconnected

<u>blocks</u> are maximal biconnected components of G

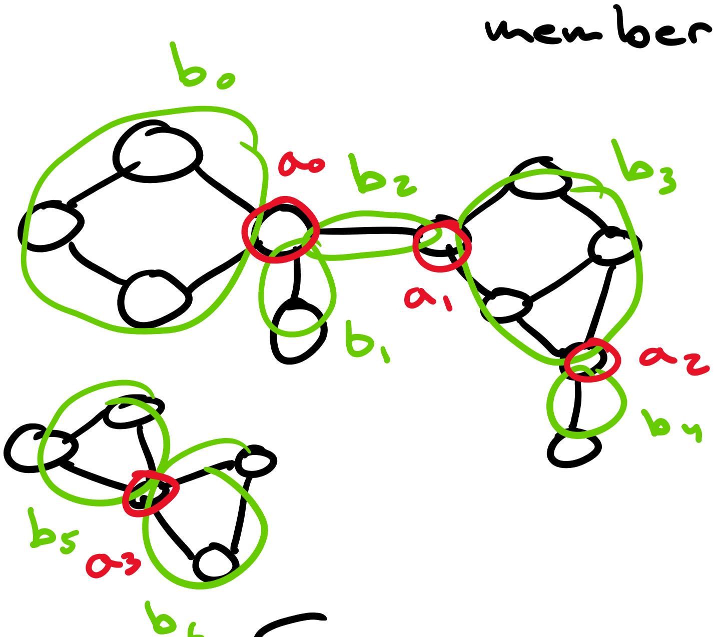
<u>articulation</u> vertices are cut vertices

<u>bridges</u> are cut edges

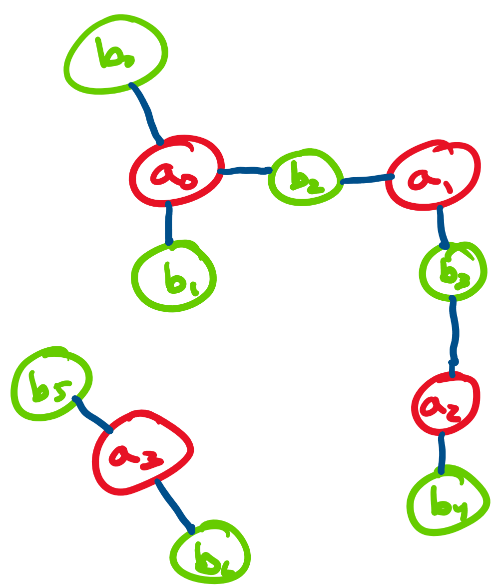
Using the above, we can construct a <u>block-cutpoint graph</u> G_B

$V(G_B) = \{$blocks & articulation verts$\}$

$E(G_B) = \{$articulation vertex membership in a block$\}$



G



G_B

Note: G_B is a tree