

## 13.1 2-Connected Graphs

We're going to talk more specifically about 2-connected and 2-edge-connected graphs. We can characterize them using **internally disjoint** and **internally edge-disjoint** paths. Two  $u, v$ -paths are internally disjoint if there is no common internal vertex. Similarly, two  $u, v$ -paths are internally edge-disjoint if there is no common internal edge. **Whitney** proved that a graph  $G$  of at least three vertices is 2-connected if and only if for all  $u, v \in V(G)$  there exists at least two internally disjoint  $u, v$ -paths. We can also prove this. Additionally and equivalently:

- $G$  is connected and has no cut vertex
- $\forall u, v \in V(G)$  there exists some cycle  $C \in G : u, v \in C$
- $\delta(G) \geq 1$  and every pair of edges in  $G$  lies on a common cycle

A **subdivision** of an edge  $(u, v)$  is the operation of replacing  $(u, v)$  with two edges attached to a new vertex, i.e.,  $(u, w)$  and  $(v, w)$ . Subdividing any arbitrary edge in a 2-connected graph will not affect the graph's 2-connectivity.

An **ear decomposition** of  $G$  is a decomposition of the edges of  $G$  into a sequence of paths  $P_0, P_1, \dots, P_k$ , where  $P_0$  is a closed path (cycle) and for  $i \geq 1$   $P_i$  has unique endpoints in  $P_0 \cup \dots \cup P_{i-1}$ . These  $P$  are called **ears** or **open ears**. A graph is 2-connected if and only if it has an ear decomposition and every cycle in a 2-connected graph is the initial cycle in some ear decomposition. We will use the idea of subdivisions in our proof of the preceding sentence.

A **closed-ear decomposition** of  $G$  is a decomposition  $P_0, \dots, P_k$  such that  $P_0$  is a cycle and  $P_i$  for  $i \geq 1$  is a path with unique or non-unique endpoints in  $P_0 \cup \dots \cup P_{i-1}$ . These  $P$  are called **closed ears**. A graph is 2-edge-connected if and only if it has a closed-ear decomposition and every cycle in a 2-edge-connected graph is the initial cycle in some closed ear decomposition.

Note that every 2-connected graph is necessarily 2-edge-connected, per our bounds discussed last lecture.

## 13.2 Biconnectivity

A graph that has no cut vertices is also called **biconnected**. We note that graphs  $K_1$  and  $K_2$  would also be considered biconnected even if they aren't 2-connected by our prior characterizations. The **biconnected components** (BiCCs) of a connected (but not necessarily biconnected) graph are the maximal subgraphs of the graph that are themselves biconnected. These are also called **blocks**. A vertex that connects to different blocks is called an **articulation point** or simply a **cut vertex**. A **block-cutpoint**

**graph** is a bipartite graph where one partite set consists of cut-vertices and one partite set consists of contracted representations of every BiCC. Edges in this bipartite graph represent which articulation points connect which blocks.