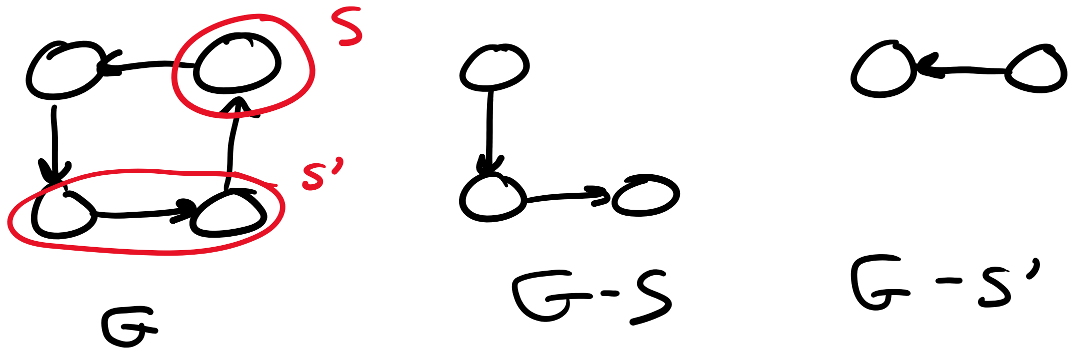
Digraph Connected

Vertex cut - a set $S \subseteq V(G)$ s.t. $G - S$ is not strongly connected



$K(G)$ = connectivity of G
$= \min |S|$ → aka size of
$\forall S \subseteq V(G)$ a minimum cut

Edge-cut - a set $F \subseteq E(G)$ that separates $V(G)$ into two vertex sets $S, \bar{S}$

$\bar{S} = V(G) - S$

The size of some cut F is the number of edges from $S \to \bar{S}$

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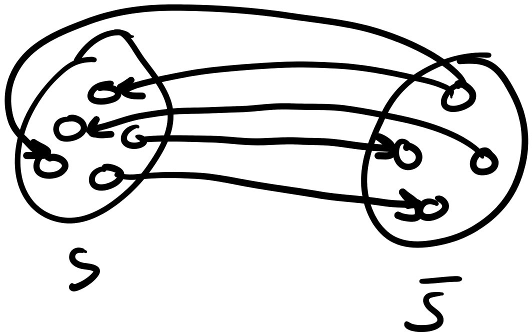

$|[S, \bar{S}]| = 2$

size of cut

$\kappa'(G)$ = edge-connectivity of G

$= \min_{S \subseteq V(G)}$ cut of S

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k-connectivity

x, y-separator - a set $S \subseteq V(G)$ s.t. G-S has no x,y-path

$\kappa(G)$ = minimum x,y-separator over all possible $x, y \in V(G)$

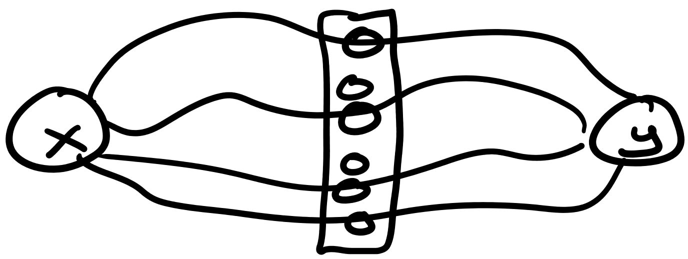
$\kappa(x,y)$ = connectivity of x, y

aka the size of a minimum x,y-separator

$\lambda(x,y)$ = maximum number of internally disjoint x,y-paths disjoint x,y-paths

First note: every x,y-separator contains a vertex from each x,y-idps

$\Rightarrow \kappa(x,y) \geq \lambda(x,y)$



Big Question

does $\lambda(x,y) \overset{?}{\geq} \kappa(x,y)$

Whitney: for $2 = \lambda(x,y) = \kappa(x,y)$ it holds

Menger: it also holds for any $k = \lambda(x,y) = \kappa(x,y)$

any $k = \lambda(x,y) = K(x,y)$
$\rightarrow$ if $(x,y) \notin E(G)$

Let's prove this using the **power** of strong induction

Induction on $|V(G)|$

Basis $P(2)$: x y

$\lambda(x,y) = 0$
$K(x,y) = 0$
$\lambda = K = 0$ ✓

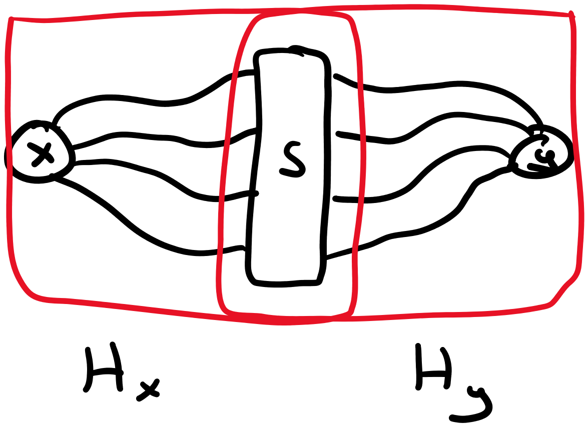
Assume we have some G
s.t. $|V(G)| = n$

Also assume for some x, y we have
$K(x,y) = k = |S| \leftarrow$ min separator

Our Goal: construct k idps given our minimum separator S

Case 1: $\exists S$ s.t. $S \neq N(x)$, $S \neq N(y)$

consider x, S-paths
consider S, y-paths

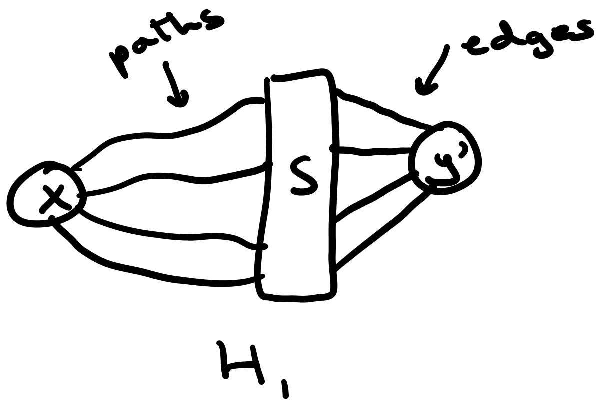


define graph

$H_1 = H_x + y'$

with edges

(s, y') $\forall s \in S$



I.H. on H_1 gives us k x, y'-idps

define H_2 as $H_y + x'$

with edges

(s, x') $\forall s \in S$

I.H. on H_2 gives us k x', y-idps

Combine the two sets of paths together to give us k idps on the original G

Case 2: $S = N(x)$ or $S = N(y)$

2a) $\exists v \notin \{x\} \cup \{y\} \cup N(x) \cup N(y)$

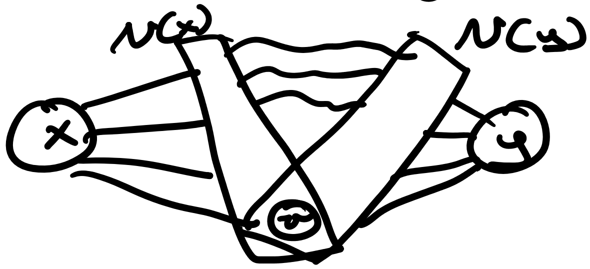
Note: v is <u>not</u> on a min cut

$\Rightarrow$ I.H. on $G-v$ gives us k x,y-idps and these paths are the same on G

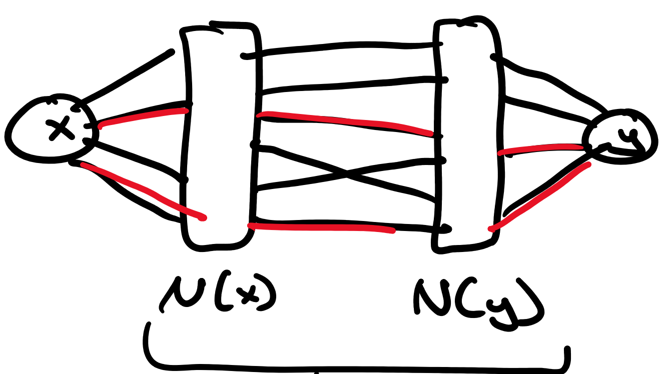
2b) $\exists v \in N(x) \cap N(y)$

$\Rightarrow$ I.H. on $G-v$ gives us $k-1$ x,y-idps

we can trivially create k paths with $(x,v),(v,y)$



2c) Otherwise, both $N(x)$ and $N(y)$ are minimum separators and there are no "external" vertices



only possible remaining configuration

Note: this is a bipartite graph

Note 2: every x,y-path uses same edge in this bipartite graph

→ we can construct idps by selecting edges as a match

|max match| = max # of idps

Q: Can we guarantee k-idps or a match of size k on this bipartite graph $G_{N(x), N(y)}$?

We note that each of N(x) and N(y) are minimum covers on $G_{N(x), N(y)}$

as $k = |N(x)| = |N(y)| = |\text{max match}|$ we have k matched edges

=> we can use to construct k-idps on our original G □

$$\implies K(x,y) = \lambda(x,y)$$

and therefore over all selections of x, y we can state that

G has connectivity k implies that between all pairs of vertices there are k idps

What about edge connectivity?

$K'(x,y)$ = x,y-edge-connectivity
= minimum x,y edge cut

$\lambda'(x,y)$ = max number of x,y edge-disjoint paths

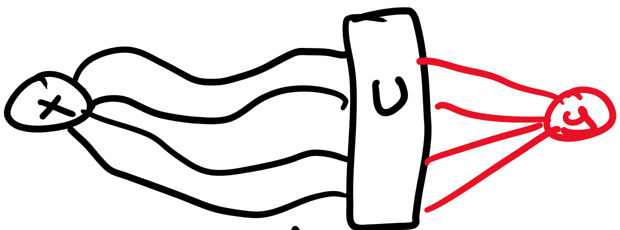
AND $K'(x,y) = \lambda'(x,y)$

Proof: can do same basic approach

$\boxed{\begin{array}{l} G \text{ is } k\text{-connected if} \\ \quad \forall x,y \in V(G):\ K(x,y) \geq k \\ \quad\quad \lambda(x,y) \geq k \\ G \text{ is } k\text{-edge-connected if} \\ \quad \forall x,y \in V(G):\ K'(x,y) \geq k \\ \quad\quad \lambda'(x,y) \geq k \end{array}}$

Let's get generalizing

<u>x, U-fan</u> - a set of paths from x to vertex set U s.t. the paths only share vertex x



internally disjoint

G is k-connected iff $|V(G)| \geq k+1$ and $\forall x \in V(G),\ U \subseteq V(G) : |U| \geq k$ $\exists$ x,U-fan of size at least k

($\Rightarrow$) Construct G' as $G+y$, where y is attached to all $u \in U$

Note: G' is also k-connected, as the addition of y cannot create a smaller vertex cut on G

$\rightarrow$ Per Menger, we have k x,y-idps

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and U is therefore a x, U-fan

(⟸) $\forall v \in V(G)$ and $U = V(G) - v$

we have v, U-fan of size k

per our assumptions

↪ $\delta(G) \geq k$

Consider some $w, z \in V(G)$

and define $U = N(z)$

As $|U| \geq k$, we have a w, U-fan

of cardinality k per assumptions

→ the edges from U to z will

give us our k-idps

→ Per Menger, G is k-connected □

We can use this result

to prove another generalization

of a result from Whitney

Recall: if G is 2-connected then

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$\forall x,y \in V(G)$ $\exists C$ where $x,y \in V(C)$

In general: if G is k-connected then

$\forall S \subseteq V(G) : |S| = k$ $\exists C$ where

$S \subseteq V(C)$

Strong induction on k for k-connected G

Basis $P(2) \rightarrow$ we showed this via Whitney

we have $P(k > 2)$ as k-connected G

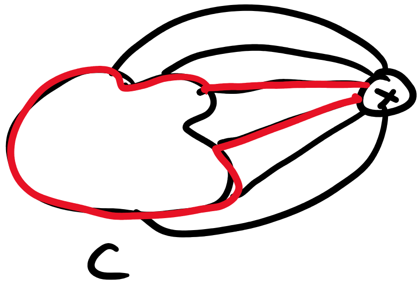
and we have some $S \subseteq V(G) : |S| = k$

via I.H. $S - \{x\}$ has some $C : |V(C)| \geq k-1$

as $P(k)$ is $(k-1)$-connected

Case 1: $|V(C)| = k-1$

we have an $x, V(C)$-fan of size $k-1$



C

→ we have some idps from x to 2 consecutive vertices in C

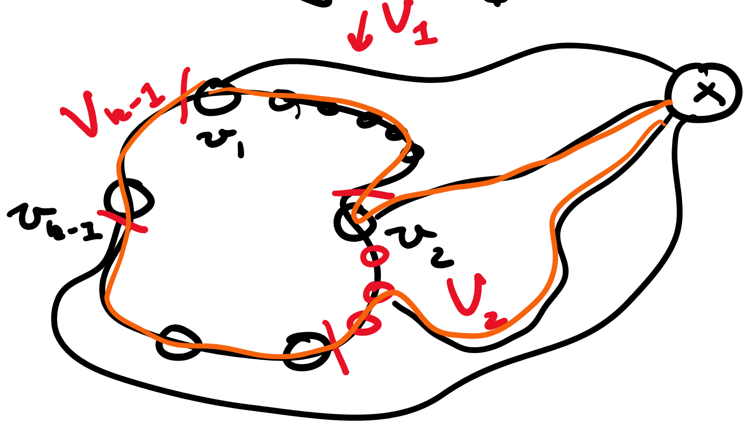
→ enlarge C to include x

Case 2: $|V(C)| \geq k$

G has $x, V(C)$-fan of size k

Order vertices of $S - \{x\}$ on C as $v_1, v_2 \ldots v_{k-1}$

Order sets of all vertices on C as V_i contains all vertices on C from v_i up to but not including v_{i+1}



→ we have k-1 disjoint sets on C, which we can use to find paths to x via that $x, V(C)$-fan

Note: for some v_i, there are two paths on the $x, V(C)$-fan

$\Rightarrow$ modify C by detouring from these two paths to x, to construct C' containing all of S $\square$