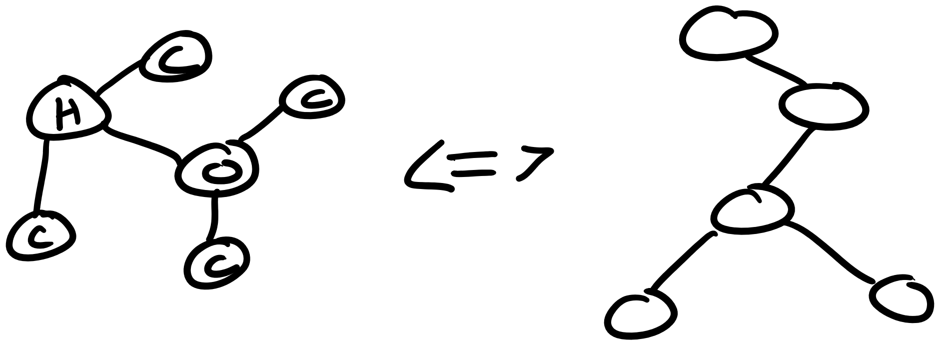
Lecture 16 - Vertex Coloring

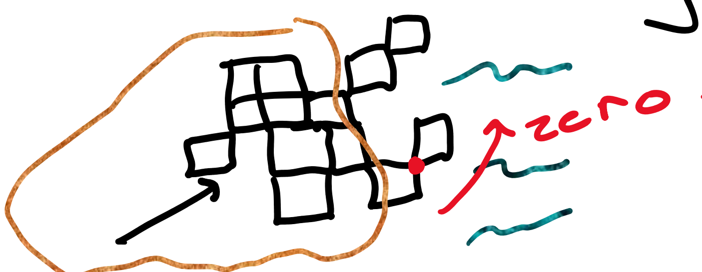
Sunday, March 17, 2024 5:08 PM

Applications

① (sub) graph isomorphism



② (bi) connectivity



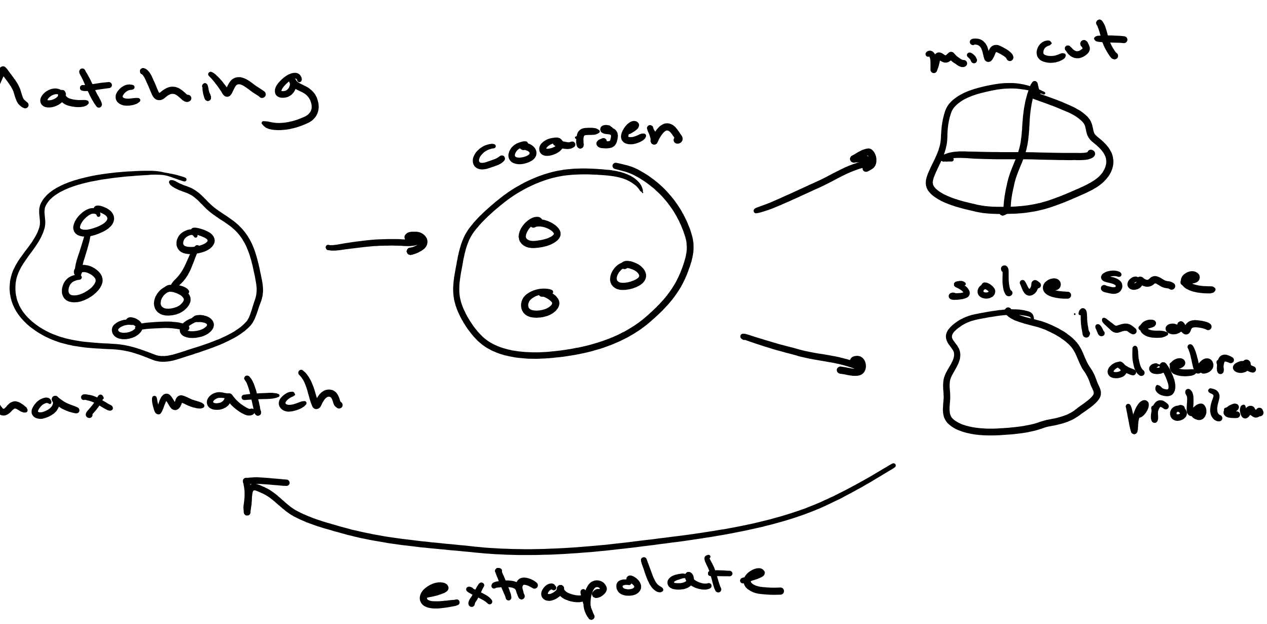
zero resistance

Ocean

Antarctic ice sheet

→ identify cut points and remove the "bad" components

③ Matching



④ Coloring

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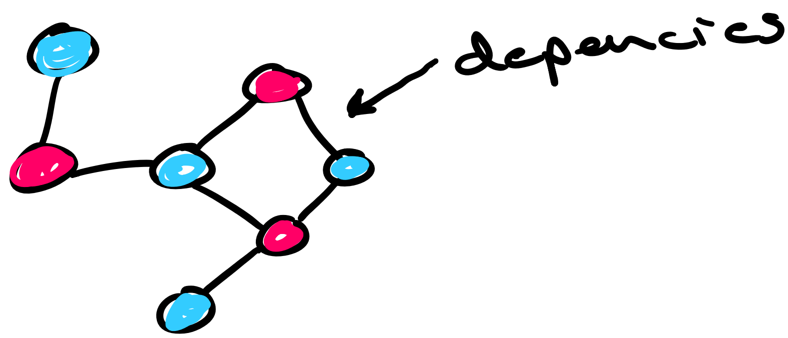


Biznix → Graph Coloring

(vertex coloring)

<u>k-coloring</u> of G is a labeling $f: V(G) \to S$, $k = |S|$ (S: set of colors)

<u>proper coloring</u> of G is a k-coloring of G s.t. no two neighbors have the same color

<u>chromatic number</u> of $G = \chi(G)$

$\chi(G)$ = the minimum k for which G is k-colorable

<u>optmal coloring</u> of G is a proper

$k = \chi(G)$

Optimal coloring of G is a proper

k-coloring for $k = \chi(G)$

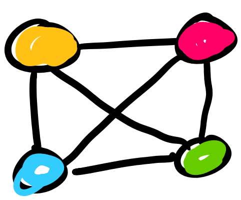
G is color-critical if for all

subgraphs $H \subset G$, $H \not\cong G$

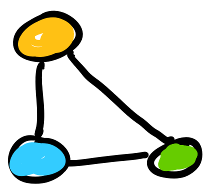
H is not iso. to G

$$\chi(H) < \chi(G)$$

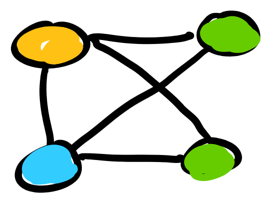
Note: all cliques are color-critical



K_4

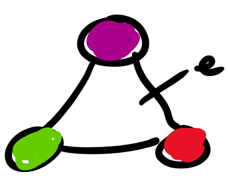


$K_4 - v$

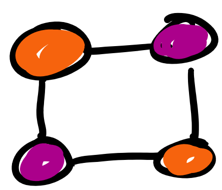


$K_4 - e$

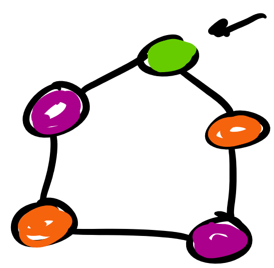
Note 2: odd cycles are color-critical



C_3



C_4



C_5

$$\chi(C_n : n = \text{odd}) = 3$$

$$\chi(C_n : n = \text{even}) = 2$$

Greedy Color Algorithm

Greedy Color Algorithm

Initialize all $v \in V(G)$ to "empty" color
for all $v \in V(G)$ in <u>some order</u>
    color v with the "least" color
    that does not appear in $N(v)$

Note: Greedy coloring is not optimal

↳ Quality depends on the vertex processing order

Let's Talk Bounds

(on G's chromatic number)
(consider G as simple)

In general for non-null G

$$1 \leq \chi(G) \leq |V(G)|$$

For non-empty G

$$2 \leq \chi(G) \leq |V(G)|$$

If G is a tree

If G is a tree

$$2 = \chi(G)$$

In general, if G is bipartite

$$2 = \chi(G)$$

If G is a clique

$$\chi(K_n) = n$$

If G is an odd cycle

$$\chi(G) = 3$$

Define $\omega(G)$ as the "clique number"

$\rightarrow$ $\omega(G)$ is the size of the largest clique $K_n \subseteq G$

For any graph

$$\chi(G) \geq \omega(G)$$

Consider our greedy algorithm

$$\chi(G) \leq \Delta(G) + 1$$

Q: Can we improve on this bound?

A: Per Brooks → yes we can

Brooks: $\chi(G) \leq \Delta(G)$

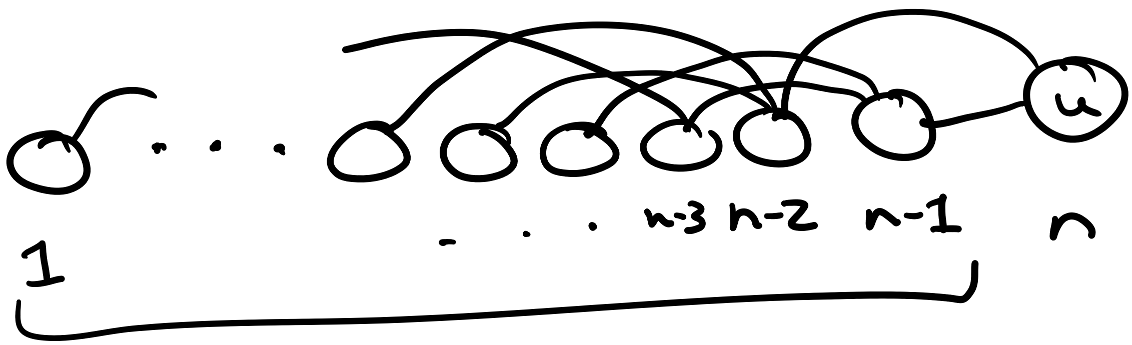
(except for cliques and odd cycles)

To prove: construct an ordering for greedy coloring s.t. we can guarantee each vertex has at most $\Delta(G)-1$ prior neighbors

Case 1: G is not k-regular

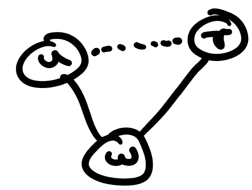
- select some $u \in V(G) : d(u) < \Delta(G)$
- grow a spanning tree from u
- apply order in reverse

→ every max-degree is guaranteed at least one higher-ordered neighbor



any $\Delta(G)$ vertex is in here

Case 2: G is k-regular



- consider $\{u, v\} \in N(w)$

  s.t. $(u, v) \notin E(G)$

To construct our order

- u, v are listed first
- w is listed last
- grow a spanning tree from w and use the reverse order

$C(u) = C(v)$

so w has at most $\Delta(G) - 1$ colors in $N(w)$

colors in $N(w)$

all will have a higher-ordered neighbor

$\Rightarrow$ At most $\Delta(G)-1$ colors will show up in $N(x)$: $\forall x \in V(G)$ $\square$

Q: How tight are these bounds?

(Consider $\Delta(G)$, $\omega(G)$, $\chi(G)$)

$$\omega(G) \overset{?}{\leq} \chi(G) \overset{?}{\leq} \Delta(G)$$

A: not very

For example:

Consider a tree T

$\Delta(T) \to \infty$

$\chi(T) = 2$

$\chi(T) <<<<<<<<<<<<< \Delta(T)$

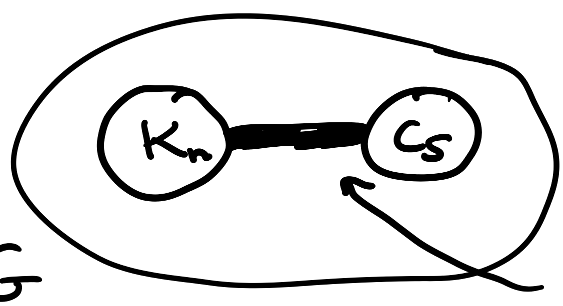
→ our upper bound can be arbitrarily loose in general

What about lower bounds?

$2 \le \chi(G)$ if G is non-empty

$\omega(G) \le \chi(G)$ more generally

First: consider some graph where we can guarantee inequality in the above relation



G

join = all edges (u,v) exist $\forall u \in V(K_n)$ $\forall v \in V(C_5)$

Note: we need $n+3$ to color G, while $\omega(G) = n+2$

→ So $\omega(G)$ can be loose

Q: How looooooooose?

A: L∞se ← infinitely

How can we show this?

→ We'll develop a construction that will increase $\chi(G)$ but will have $\omega(G)$ remain fixed

AKA <u>Mycielski's Construction</u>

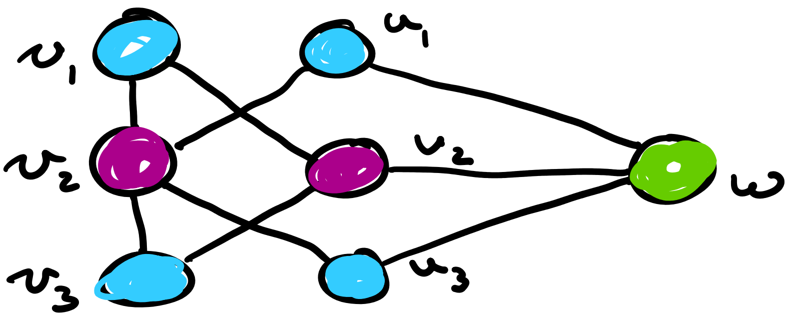
Given a triangle-free G with $\chi(G) = k$, we can construct G' with $\chi(G') = k+1$ and $\omega(G) = \omega(G') = 2$

consider: $v_1 v_2 \dots v_n \in V(G)$

create: $u_1 u_2 \dots u_n$

add edges between u_i and all $v_j \in N(v_i)$

Create vertex w and add edges from w to all u_i



G $\quad$ G'

$\chi(G) = 2$

$\omega(G) = 2$

$\chi(G') = \chi(G) + 1 = 3$

$\omega(G') = \chi(G) = 2$

Note: we don't create any triangles

Note 2: coloring all of u_i requires the same # colors for all of v_i

Note 3: coloring w require a separate new color

Note 4: we can iterate this construction infinite times

$$\Rightarrow 2 = \omega(G^{''''''\times\infty}) <<<<<<<<< \chi(G^{'''''\times\infty})$$

Takeaway: our bounds tell us very little in the general case

¯_(ツ)_/¯

Next time: minimum vertex coloring

→ very computationally hard

→ so is getting $\chi(G)$ $\square$