16.1 Vertex Coloring

Assume all graphs are undirected, connected, and without self loops. You'll note that multi-edges are irrelevant for the following discussions and self loops make the problems undefined.

A k-**coloring** of a graph is a labeling $f : V(G) \to S$, with $k = |S|$. Essentially, each vertex is assigned a **color** of $1 \ldots k$. A coloring is **proper** if no adjacent vertices share the same color. A graph is k-**colorable** if it has a proper k-coloring.

The **chromatic number** of a graph $\chi(G)$ is the least k for which G is k-colorable. A graph is k-**chromatic** if $\chi(G) = k$. A proper k-coloring of a k-chromatic graph is an **optimal coloring**. Note that an optimal coloring is not necessarily unique.

If $\chi(H) < \chi(G) = k$ for every subgraph H of G, where H is not isomorphic to G, then G is **color-critical**.

Remember back to when we discussed independent sets. We can observe that sets comprised of vertices of the same color in a proper coloring are all independent.

16.2 Greedy Coloring

A simple greedy algorithm for creating a proper coloring is shown below. The basic idea is do a single pass through all vertices of the graph in some order and label each one with a numeric identifier. A vertex will labeled/colored with the lowest value that doesn't appear among previously colored neighbors.

```
procedure GreedyColoring(Graph G(V, E))
    for all v ∈ V(G) do
        color(v) ← −1
    for all v ∈ V(G) in order do
        isCol(1 . . . Δ(G) + 1) ← false
        for all u ∈ N(v) where color(u) ≠ −1 do
            isCol(color(u)) ← true
        for k = 1 . . . Δ(G) + 1 do
            if isCol(k) = false then
                color(v) ← k
                break
    k ← max(color(1 . . . n))
    return k
```

16.3 Coloring Bounds

Obviously, $1 \leq \chi(G) \leq n$.

For a graph with a non-empty edge set, its easy to see that $2 \leq \chi(G)$.

For a tree, or any other bipartite graph, we can show that $2 = \chi(G)$.

For a clique $K_n : \chi(G) = n$. The clique number of G, $\omega(G)$, is the maximum size of any clique in a general graph G. We can see that $\chi(G) \geq \omega(G)$.

Remember that the independence number, $\alpha(G)$, is the size of the largest possible independent set. We can also see that $\chi(G) \geq \frac{n}{\alpha(G)}$.

If $\Delta(G)$ is the maximal degree in a graph, then a logical argument based on our greedy coloring algorithm shows that $\chi(G) \leq \Delta(G) + 1$.

Running our greedy coloring algorithm on a selected vertex order can slightly improve our bounds. If we run on vertices in non-increasing order, or $d_1 \geq d_2 \geq \cdots \geq d_n$, then $\chi(G) \leq 1 + \max\limits_{i=1 \ldots n} \min\{d_i, i-1\}$. This gives a bound of one plus the maximum over all vertices of the lesser of either the vertex's degree or its index in the ordering minus one. Note that this bound will be equivalent to the maximum degree based bound discussed last class ($\chi(G) \leq \Delta(G) + 1$), should there be more than $\Delta(G)$ vertices with a degree of $\Delta(G)$.

However, we can improve this bound slightly further by creating an even smarter vertex order. If we are able to create a vertex order such that each vertex has at most $k-1 : k = \Delta(G)$ lower-indexed neighbors, then greedy coloring will give us a bound $\chi(G) \leq \Delta(G)$. **Brooks' Theorem** states we can create such an order for all graphs except cliques and cycle graphs with an odd length.

16.4 Looseness of these Bounds

In practice, the above bounds are very loose. Trees have a chromatic number of $\chi(T) = 2$, but can have an arbitrarily large maximum degree. Likewise, we've noted before that $\chi(G) \geq \omega(G)$, where $\omega(G)$ is the size of the largest clique in G.

However,triangle-free graphs can have an arbitrarily large chromatic number. One way to demonstrate this is to create triangle-free graph through **Mycielski's Construction**. Given triangle-free graph G with vertices $\{v_1, \ldots, v_n\}$ and $\chi(G) = k$, we can create G' with $\chi(G') = k+1$ by adding vertices $\{u_1, \ldots, u_n\}$ and vertex w to G, making u_i adjacent to all in $N(v_i)$ and having $N(w) = \{u_1, \ldots, u_n\}$. An examination of this construction shows that it produces a triangle-free graph G' with a larger chromatic number than G.