Lecture 17 - More coloring

Thursday, March 21, 2024 10:01 AM

Properties of k-chromatic graphs

How small can a k-chromatic graph be?

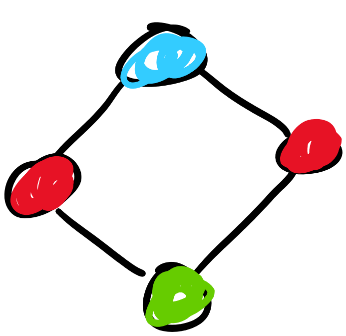
Consider all possible color pairs

$\binom{k}{2}$ total number combinations on any given edge

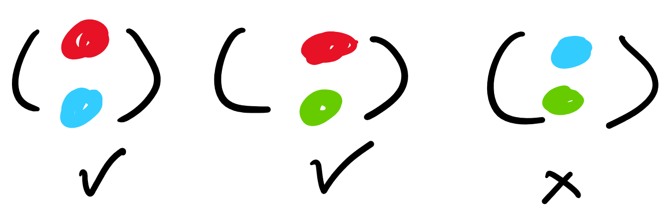
→ also the minimum number of edges on k-chromatic graph

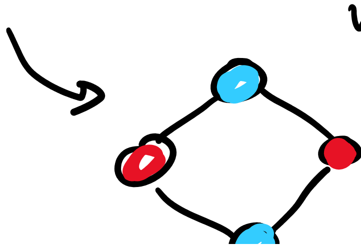
whY?

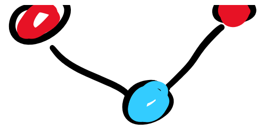
Every combination must exist, as otherwise we could combine colors to get a $<k$-coloring



Possible color combos



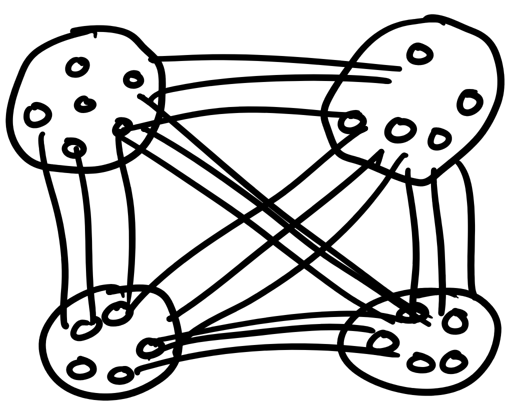




=> any k-chromatic graph must have at least $\binom{k}{2}$ edges

What about how BIG?

Think in terms of a generalization of bipartite graphs → multipartite graph (k-partite)



To maximize: make a complete multi-partite graph

Can we further maximize for some given k, $|V(G)|$?

→ set all partite sets to be equal in size (± 1)

=> Turán graph

$\Rightarrow$ Turán graph

Q: Does it maximize $|E(G)|$?

- Consider an "unbalanced" complete multipartite graph

$\exists S_i S_j$ s.t. $|S_i| + 1 < |S_j|$

move $v \in S_j$ to S_i

$\rightarrow$ edges lost $= |S_i|$

$\rightarrow$ edges gained $= |S_j| - 1$

$\rightarrow$ as $|S_j| > |S_i| + 1$

we have a net gain on $|E(G)|$

If we repeat this process, we will eventually maximize $|E(G)|$

$\Rightarrow$ Turán graph is the largest possible k-chromatic graph on $|V(G)|$ vertices $\square$

graph on $|V(G)|$ vertices $\square$

Color-critical graphs

G is color-critical if

$\forall v \in V(G) : \chi(G-v) < \chi(G)$

$\forall e \in E(G) : \chi(G-e) < \chi(G)$

G is <u>k-critical</u> if the above hold and $\chi(G) = k$

For k-color-critical graph G:

$\exists$ some k-coloring on G s.t.

<u>$\forall v \in V(G)$</u> the color $c(v)$ appears nowhere else and there are $k-1$ colors in $N(v)$

→ consider $(k-1)$-coloring on $G-v$

If we add back v and not all $(k-1)$ colors show up in its neighborhood its neighborhood

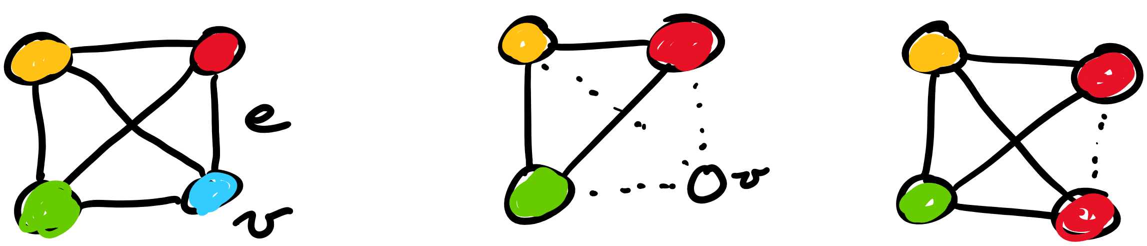
↳ we'd have a $(k-1)$-coloring on G, just assign one of missing $(k-1)$ colors to v

contradiction

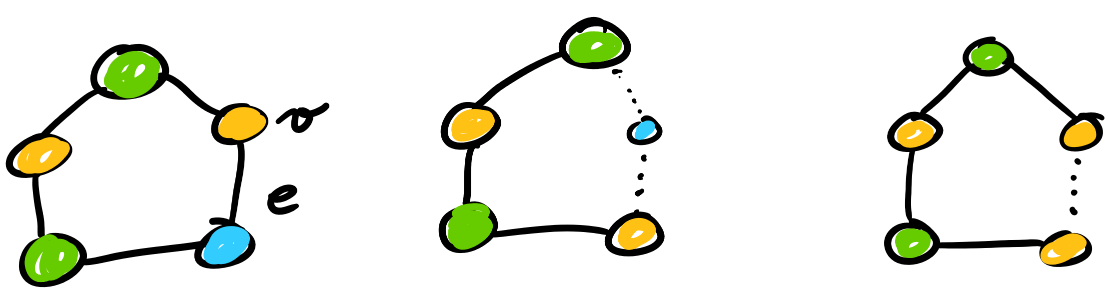
Similarly: $\forall e = (u, v) \in E(G)$

→ Every proper $(k-1)$-coloring on $G - e$ gives $c(u) = c(v)$

→ If not, we would have a $(k-1)$-coloring on G



G $\quad$ $G - v$ $\quad$ $G - e$



G $\quad$ $G - v$ $\quad$ $G\,e$

$G \qquad G-v \qquad G_e$

Connectivity of k-color-critical graph G

show: G is (k-1)-edge-connected

To do so, first show:

For G' s.t. $\chi(G') > k$, let $\{X, Y\}$ be a paritition of $V(G')$

If $G'[X]$ and $G'[Y]$ are both k-colorable $|[X,Y]| \geq k$

induced subgraph of G' on $X \subseteq V(G')$

consider $X_1, X_2 \dots X_k$ and $Y_1, Y_2 \dots Y_k$

as independent sets defined by our assumed k-coloring

show: if $|[X,Y]| < k$, $\exists X_i Y_j$ that we can combine to form

we can combine to form
a k-coloring on G'

Assume $|[X,Y]| < k$

construct H as bigraph

$V(H) = \{$ each $X_i\ Y_j$ coarsened to a single vertex $\}$

$E(H) = \{ (X_i Y_j)$ for all i,j pair where NO edge exists between some $x \in X_i,\ y \in Y_j$ on original $G' \}$

Note: H has more than $k(k-1)$ edges

→ k^2 possible, but cut $< k$

Note x2: m vertices cover at most $m * k$ edges in H

$\rightarrow$ $E(H)$ is not covered by only $(k-1)$ vertices

$\Rightarrow$ min cover $\geq k$

max match $\leq k$

min cover $=$ max match $= k$

If we combine all matched sets into a single color

$\Rightarrow$ we get a k-coloring on G'

contradiction

$\Rightarrow |[X,Y]| \geq k$

Bring it on home

$\rightarrow$ show every k-critical graph is $(k-1)$-edge-connected

Consider k-color-critical G

$[X,Y]$ defines some min cut

$\rightarrow G[X]$ and $G[Y]$ are $(k-1)$-colorable

$\Rightarrow$ edge cut must be at least $(k-1)$ in size $\square$

More on properties of G

<u>S-lobe</u> $\rightarrow$ given $S \subseteq V(G)$ and $G-S$, an S-lobe is an induced subgraph on S + (some component of $G-S$)

If G is k-critical, G has no min vertex cut $\{x,y\}$ where $(x,y) \in E(G)$ AND if G has vertex cut $S = \{x,y\}$ then G has an S-lobe H where $\chi(H + (x,y)) = k$

Assume we have cut S

Assume we have cut S

and $G-S \to H_1 H_2 \dots H_t$ as S-lobes

$\to$ each H_i are $(k-1)$-colorable

consider $\{x, y\}$ and $c(x), c(y)$ on H_i

$\to$ on same H_i: $c(x) \neq c(y)$ otherwise we can create a $(k-1)$-coloring on G

$\hookrightarrow$ we can combine all $(k-1)$-colored S-lobes, as we have no dependence between them except for the x, y

Note: this same logic implies at least on one H_i that $c(x) = c(y)$

$\to$ so x, y must not be adjacent

Consider H_j that has $c(x) = c(y)$

$\to$ adding edge (x, y) requires a new color for x or y

$\Rightarrow \chi(H+(x,y)) = k \quad \square$

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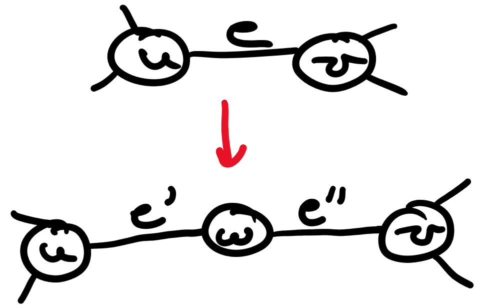
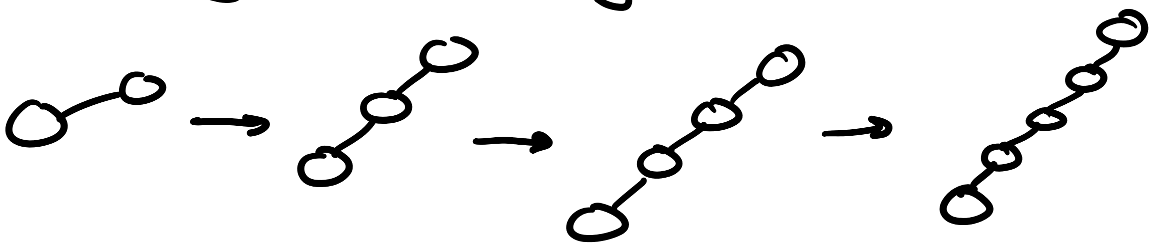
subdivisions and coloring

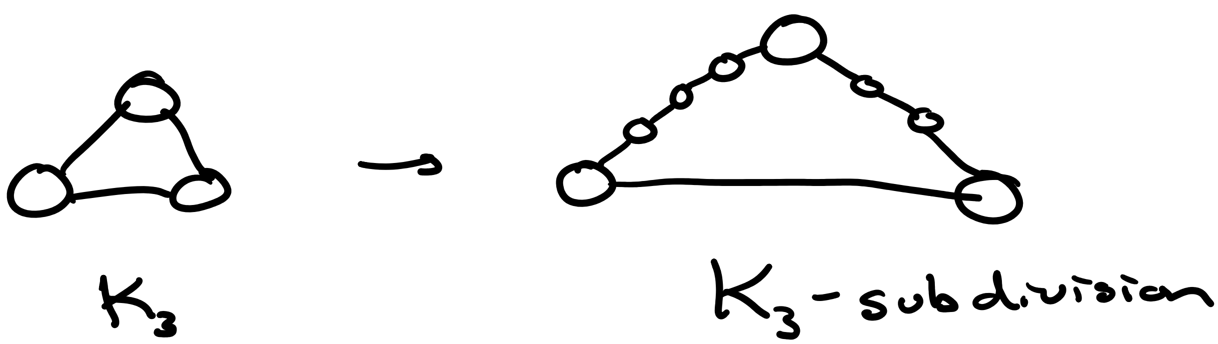
Recall: subdivision



Note: we can subdivide an edge any arbitrary number of times



Also note: we can also subdivide edges of some (sub)graph G to create a G-subdivision



Basically: we replace edges with paths in our G-subdivision

Prove: Every graph G where $\chi(G) \geq 4$ has a K_4-subdivision

We'll do strong induction on $|V(G)|$

Basis $P(4) = K_4$ and trivial

$P(n > 4)$ we have some G s.t. $\chi(G) \geq 4$

→ consider some 4-critical subgraph of G

(i.e. some H s.t. $\chi(H) = 4$ and $\chi(H - e) = 3$ $\forall e \in E(H)$)

Case 1: H is 2-connected

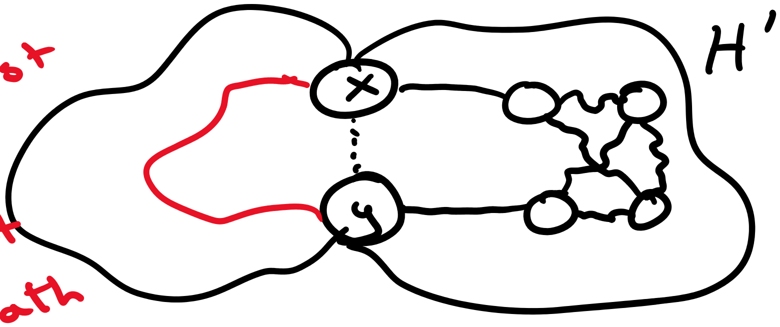
↳ Assume we have $S = \{x, y\}$ where $(x, y) \notin E(G)$

Consider S-lobe H' of H

Consider S-lobe H' of H
where $\chi(H' + (x,y)) \geq 4$

We take $H' + (x,y)$ as $P(k)$ and
get a K_4-subdivision via I.H.

we just need to show that $\exists$ x,y-path



→ trivial → as the S-lobes of G are connected and we have at least 2

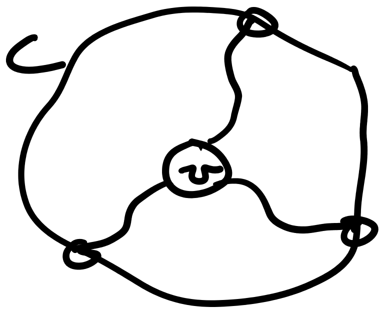
Recall: S is a minimum cut, which separates G into at least 2 components

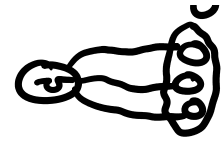
Case 2: H is at least 3-connected

Consider $H - v$, $v \in V(H)$

→ as $H - v$ is at least 2-connected we have a cycle of at least length 3

least length 3





Recall: v, U-fans

→ a k-connected graph has a v,U-fan where $|U| \geq k$

→ we consider v, C-fan, which has at least 3 internally disjoint paths from v

→ we can use those paths to construct a K_4-subdivision along with C □

=> Any G with $\chi(G) \geq 4$ will contain K_4-subdivision ✓