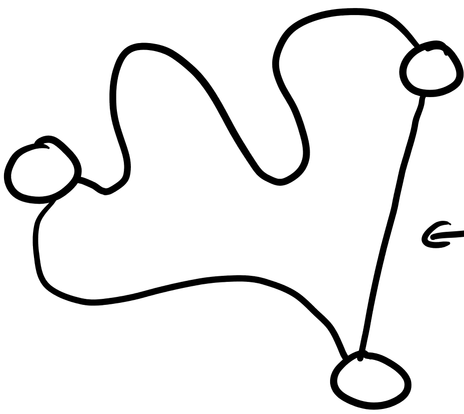
Lecture 19 - Planarity

Wednesday, March 27, 2024 6:03 PM

Planarity and Drawing

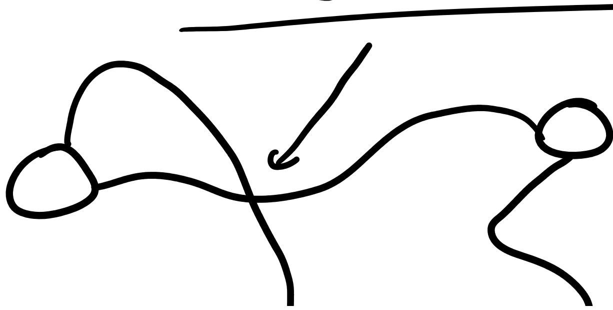
Graph drawing: a mapping of vertices to points on the plane, and edges to some curves between those points



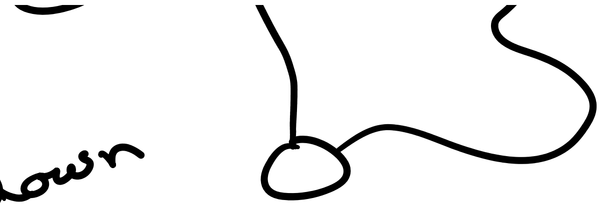
← points on the plane

← curves between points

Graph planarity: a graph is planar if it can be drawn without any edge crossing



an intersection of two edge



of two edge curves

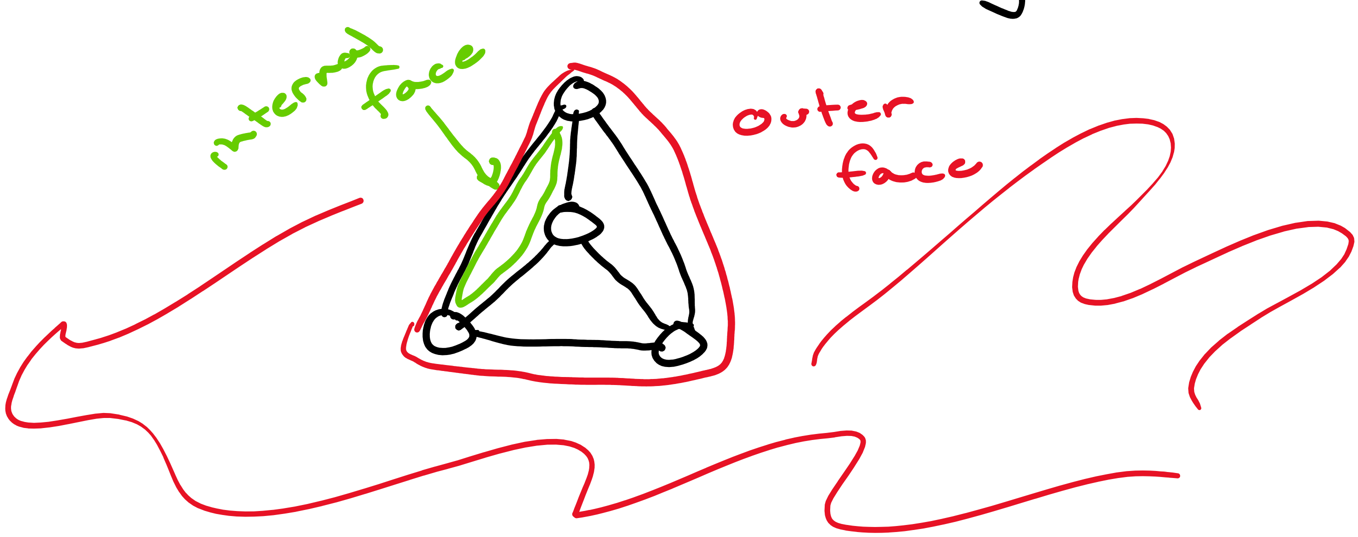
Slop down

Couple more definitions:

planar embedding: graph drawing with no edge crossings

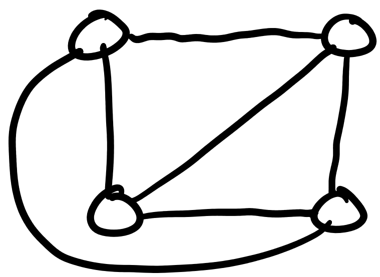
face: a maximal area in some embedding fully enclosed in edge curves

outer face: the external or unbounded face of some embedding



Note: K_4 is planar

↳ can we draw it s.t. all vertices are on the outer face?

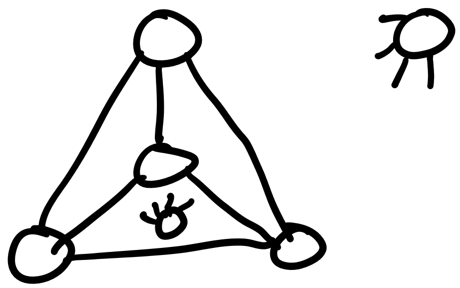


→ No

outerplanar: a graph with some embedding where all vertices are on the outerface

Q: What does that mean for K_5? (given a K_4 embedding)

→ we need an additional vertex and curves to all other vertices in K_4



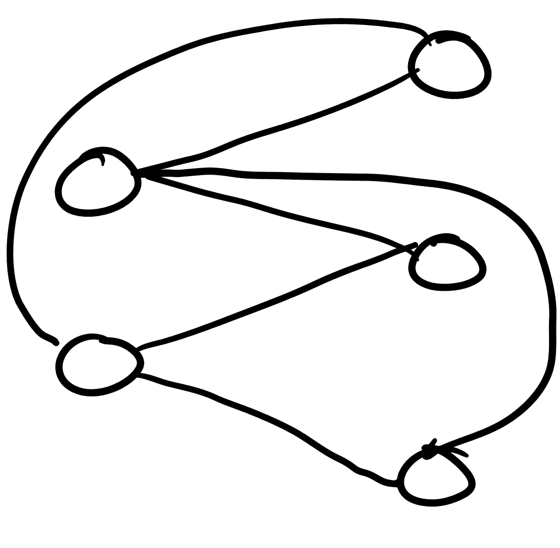
=> we cannot draw K_5 without edge crossings

so K_5 is <u>not planar</u>

aka <u>nonplanar</u>

→ a nonplanar G has no planar embedding

Also think about $K_{2,3}$ & $K_{3,3}$



$K_{2,3}$ is not outerplanar

=> so $K_{3,3}$ is nonplanar

Special subgraphs: $K_4, K_5, K_{2,3}, K_{3,3}$
(more later)

(more later)

Other examples:

C_n: are planar and outerplanar

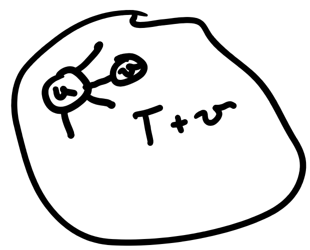
K_3: also planar and outerplanar

P_n: same

$K_{n\geq 5}$: nonplanar

Trees: planar & outerplanar

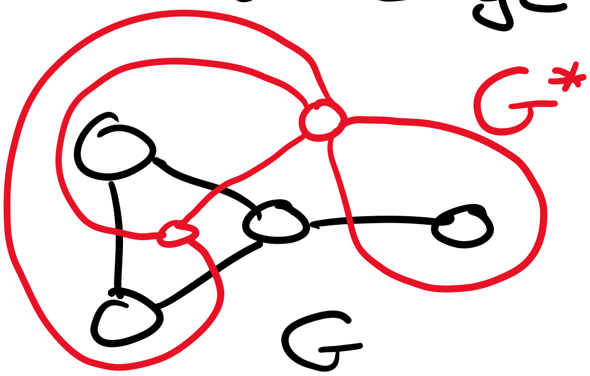
WP:10 Think in terms of weak induction

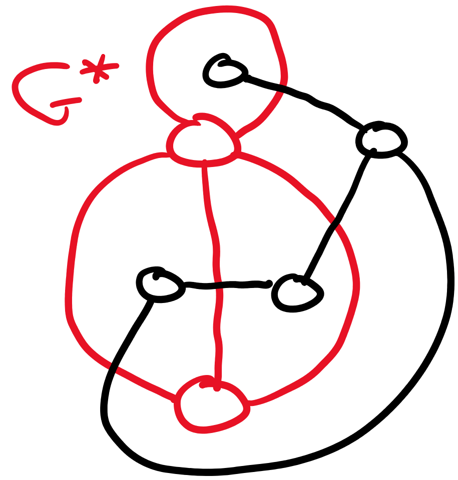


Dual Graphs

Dual graph G^* of some embedding of planar G is a graph whose vertices are the faces of G and edges are defined between

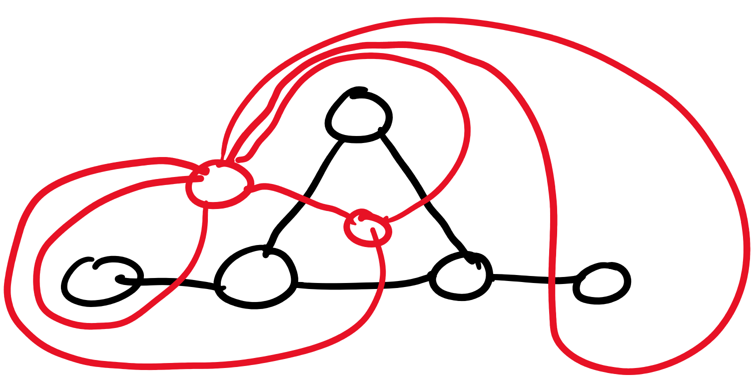
vertices are the faces of G.
and edges are defined between
the faces of G that share
an edge

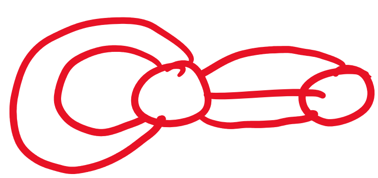


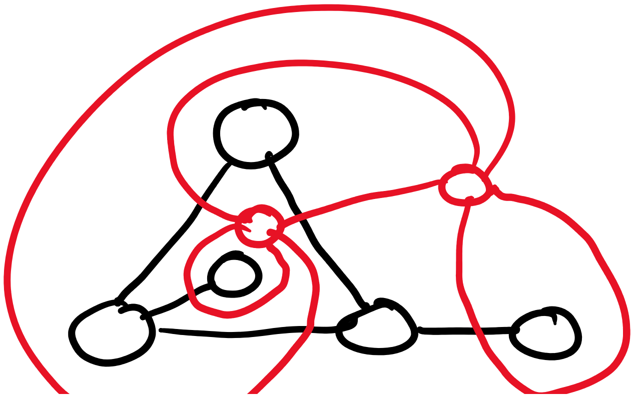


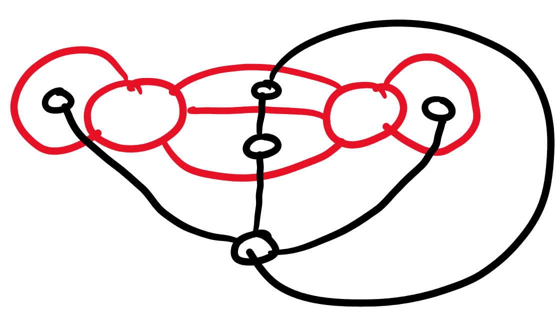
$(G^*)^* = G$
(sometimes)

Note: G^* depends on a specific
embedding of G

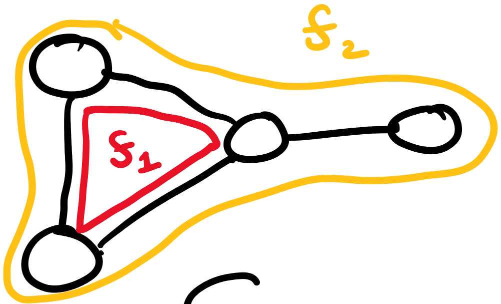








More on faces



G

G has 2 faces

The <u>length</u> of a face is equal to the number of edge curves bounding it

$\ell(f_1) = 3$

$\ell(f_2) = 5$

length

Note: each edge contributes +2 to the sum of face lengths for some G

$$\sum_i \ell(f_i) = 2|E(G)|$$

Proofy business

planar

planar

G is bipartite $\Leftrightarrow$ all faces in an embedding are even length

$\Leftrightarrow G^*$ is Eulerian

G is bipartite $\Rightarrow$ all faces even

Note: all possible closed walks are even

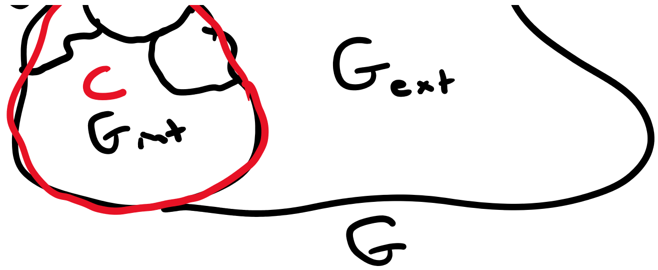
→ face lengths are defined by a closed walk

$\Rightarrow$ all faces are even

All faces even $\Rightarrow G$ is bipartite

Consider some cycle in G

→ all of G is either internal or external to that cycle in some embedding of G



Consider the internal portion of G

→ all faces are even

$$\sum_{i \in G_{int}} \ell(f_i) = \text{even}$$

Note: each internal edge is counted twice

Note 2: each edge on C is counted once

parity

⇒ the cycle C must be even

and applies to any C

⇒ G is bipartite □

proovin'

All faces even ⇔ G^* is Eulerian

(⇒) Note that the degrees of

(⇒) Note that the degrees of vertices in G^* are equal to the lengths of faces in G

↳ so G^* is Eulerian

(⇐) Same logic, as G^* is Eulerian this implies even degrees which implies even faces in G $\square$

Euler's Formula

(for a planar embedding)

$$n - e + f = 2$$

n = $|V(G)|$, e = $|E(G)|$, f = # of faces in G's embedding ← planar and connected

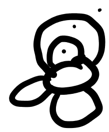
We shall prove this via the

Power

Power

of induction on $|V(G)|$

Basis: $P(1) \rightarrow$ o K_1 $\quad n=1,\ e=0,\ f=1 \quad 1-0+1=2$ ✓



$n=1,\ e=e,\ f=e+1 \quad 1-\underline{e+e}+1=2$ ✓

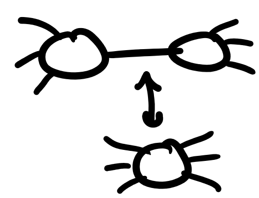
Consider our $P(n)$ case

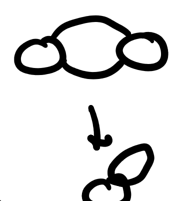
→ there exists some edge that is not a self loop

→ Contract that edge to get $P(k)$

I.H. on $P(k)$

↳ $n'-e'+f'=2$



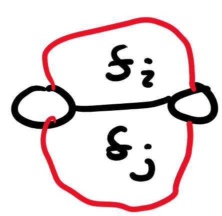


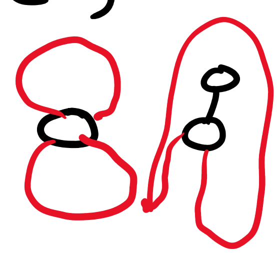
Bring it on back to $P(n)$

$n = n'+1$

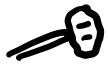
$e = e'+1$

$f = f'$ → consider how a





$f = f'$ → consider how a face is modified via edge contraction




Plug n' chug

$$n' - e' + f' = 2$$

$$(n - \cancel{1}) - (e - \cancel{1}) + f = 2$$

$$n - e + f = 2 \quad \square$$

QED

Let's put EF to use
Euler's Formula

If G is a simple connected planar graph with $|V(G)| \geq 3$

$$\Rightarrow e \leq 3n - 6$$

G is simple → $\ell(f_i) \geq 3$

From our face length sum formula

$$2e = \sum \ell(f_i) \geq 3f$$

$$2e \geq \boxed{3f}$$

Consider $n - e + f = 2$

$$(e = n + f - 2) \cdot 3$$

$$3e = 3n + 3f - 6$$

$$3e \leq 3n + 2e - 6$$

$$-2e \qquad\qquad -2e$$

$$\boxed{e \leq 3n - 6}$$

This gives us an upper bound on $|E(G)|$ w.r.t. $|V(G)|$ for a graph to be planar

What if G is triangle-free?

$$\ell(f_i) \geq 4$$

plug n' chug

...

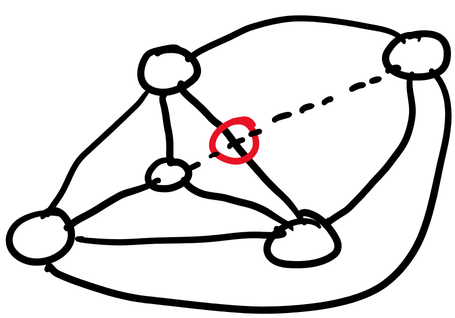
$$\boxed{e \leq 2n - 4}$$

Note: these conditions are **necessary** but NOT **sufficient**

Let's get **extreme** 

maximal planar G: adding an edge to planar G makes it nonplanar

minimal nonplanar G: deleting any edge from nonplanar G makes it planar



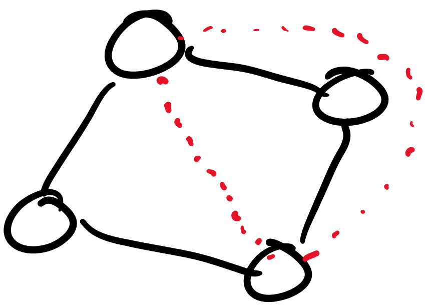
K_5 is minimal nonplanar

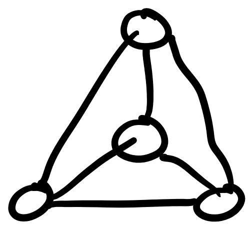
triangulation: a planar embedding where all faces are of length 3

$\boxed{\text{maximal planar} \Leftrightarrow \text{triangulation}}$

note bound $e \le 3n-6$

Also note: an edge can trivially drawn in any face of length at least 4 without crossing





K_4 is a triangulation $\Rightarrow$ maximal planar

Next class: Kuratowski's Theorem