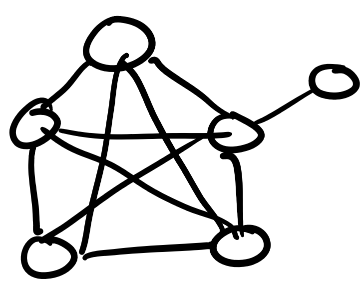
So far...

we've identified some **necessary** conditions for planarity (simple connected)

- $e \le 3n-6$
- $e \le 2n-4$ if G is triangle free

- G has no K_5 subdivision
- G has no $K_{3,3}$ subdivision

Not sufficient



$\Rightarrow$ $e = 11$, $n = 6$

$11 \le 3 \cdot 6 - 6 = 12$

Are these conditions sufficient?

Kuratowski: Yes

K_5 & $K_{3,3}$ subdivisions

→ Kuratowski subgraphs (KS.)

Note: If subgraph H is nonplanar, then any subdivision of H is also nonplanar

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Kuratowski's Theorem

G is planar iff G has no K.S.

(⇒) pretty trivial

(⇐) Buckle up (does there exist a counter-example)

① For every face f_i of a planar embedding of G, $\exists$ an embedding where f_i is on the outer face

→ any edge $e \in E(G)$ or $v \in V(G)$ can be drawn on the outer face

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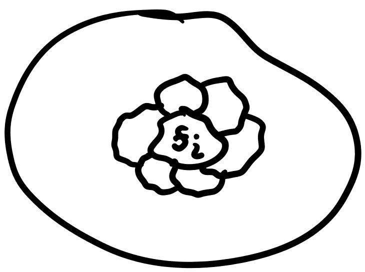
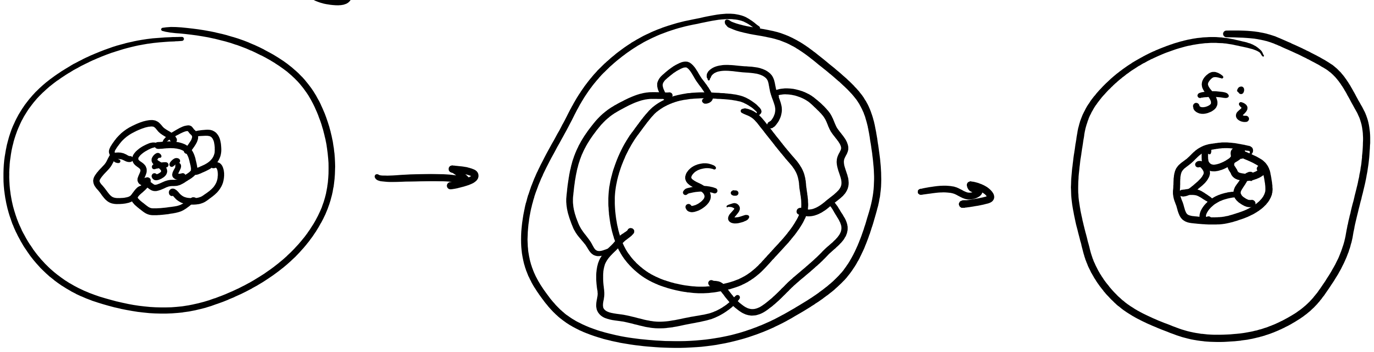
Consider an embedding of G on a sufficiently large sphere



→ expand f_i and return a projection of G bounded by f_i



② Every minimal nonplanar graph G is 2-connected

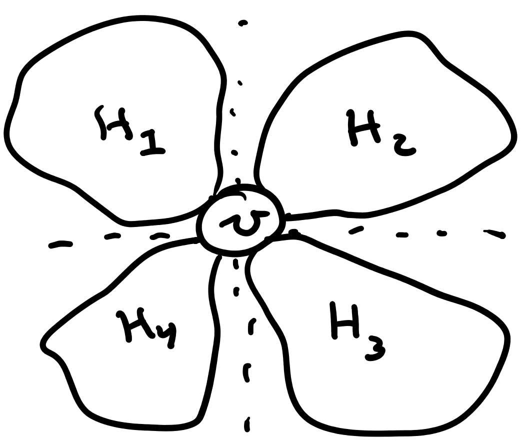
→ $\forall H \subset G$, H is planar

Assume $\exists v \in V(G)$: $G - v$ is disconnected

$$G - v = \underbrace{H_1 H_2 \ldots H_k}_{\substack{\text{components} \\ \text{of } G-v}} \quad (\text{all } H_i \text{ is planar})$$

We can create an embedding of G by "squeezing" all of H_i into $\frac{360^\circ}{k}$ around v

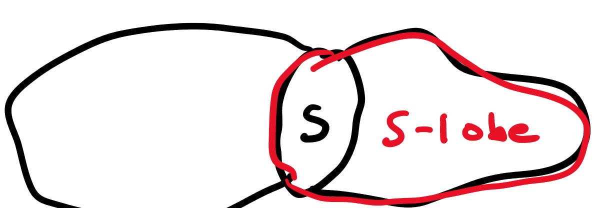
Note: From (1) for all H_i there exists an embedding with v on the outer face

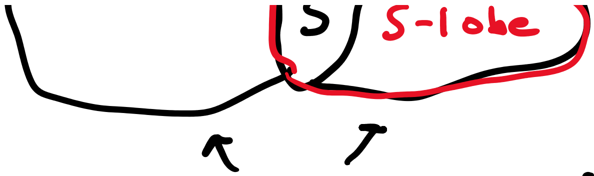


contradiction on selection of 1-connected G

$\Rightarrow G$ must be 2-connected

(3) Recall: S-lobes are an induced subgraph on some vertex cut S and some component of $G-S$





Let $S=\{x,y\}$ be a separating set of some 2-connected G

If G is nonplanar $\Rightarrow$ adding edge (x,y) to some S-lobe of G yields a nonplanar graph

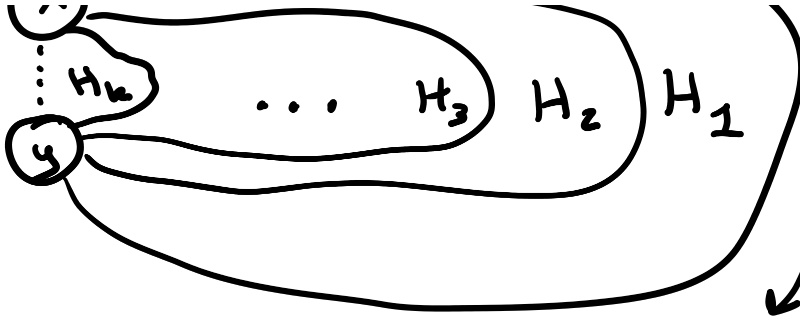
define $H_i = G_i \cup \{x,y\} \cup (x,y)$

$$\underbrace{H_i}_{\text{S-lobe}} = \underbrace{G_i}_{\text{component of } G-S} \cup \underbrace{\{x,y\}}_{S} \cup \underbrace{(x,y)}_{\text{edge}}$$

From (1), H_i has an embedding where (x,y) is on the outer face

Assume H_i is planar

→ we can iteratively embed all $H_{i=2..k}$ within the internal face of H_{i-1} containing edge (x,y)



contradiction

we have a planar embedding of G

$\Rightarrow$ at least one H_i is nonplanar

(4) If G is a graph with the fewest edge among all nonplanar graphs without a K.S., then G must be 3-connected

Note: G doesn't exist, but if it did, it would be 3-connected

Why: restrict any possible counter-example of Kuratowski to 3-connected

Note 2: deleting an edge cannot create a K.S.

$\rightarrow \forall e \in E(G): G - e$ is planar and does not have a K.S.

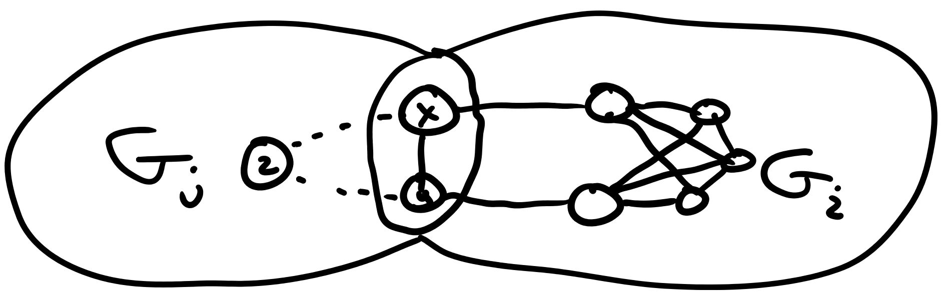
From ②, G is 2-connected

Assume $\exists S = \{x, y\}$ then some S-lobe of $G_i + S$ is nonplanar (From ③)

define H as that S-lobe $+ (x,y)$

From our minimality condition:
as $|E(H)| < |E(G)|$, H must have a K.S.

Consider the above configuration



H

However, since G is 2-connected,

$\exists z \in G_j : G_i \neq G_j$, where we have 2 disjoint z,x- and z,y-paths

(recall our fan theorem)

$\Rightarrow$ we still have a K.S. in G

when we remove edge (x, y)

contradiction

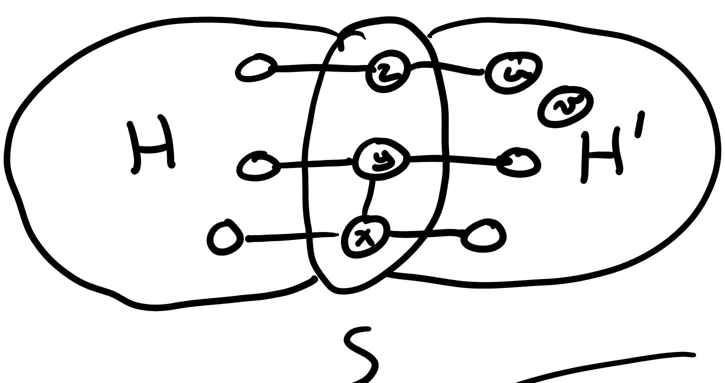
$\Rightarrow$ Any minimal counter-example must be 3-connected

Next up: show all 3-connected graphs w/o a K.S are planar

---

(5) If G is 3-connected and $|V(G)| \geq 5$, $\exists e \in E(G)$ s.t. $G \cdot e$ is also 3-connected

Consider $e = (x, y) \in E(G)$ s.t. $G \cdot e$ is not 3-connected



First: assume there does not exist any edge e s.t. $G \cdot e$ is 3-connected

↳ all edges are within same separator with same 'mate' vertex (z in the example)

Select $S = \{x, y, z\}$ s.t. $|V(H)|$ is maximum

Each of x, y, z have a neighbor in each of H and H'

- consider $u \in N(z)$, $u \in V(H')$
- consider v, the 'mate' of edge (u, z)

$G - \{z, u, v\}$ is disconnected

$V(H) \cup \{x, y\}$ is connected and within a component of $G - \{z, u, v\}$

contradiction

↳ on our selection of H

$\Rightarrow \exists e \in E(G)$ s.t. $G \cdot e$ is 3-connected

⑥ If G has no K.S.

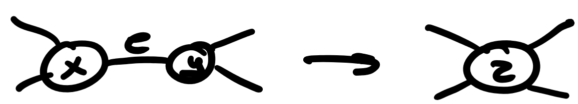
$\Rightarrow G \cdot e$ has no K.S.

Contrapositive

$G \cdot e$ has a K.S. $\Rightarrow$ G has a K.S.

define H as K.S. within $G \cdot e$

define $z \in V(G \cdot e)$, $z \leftarrow e = (x, y)$

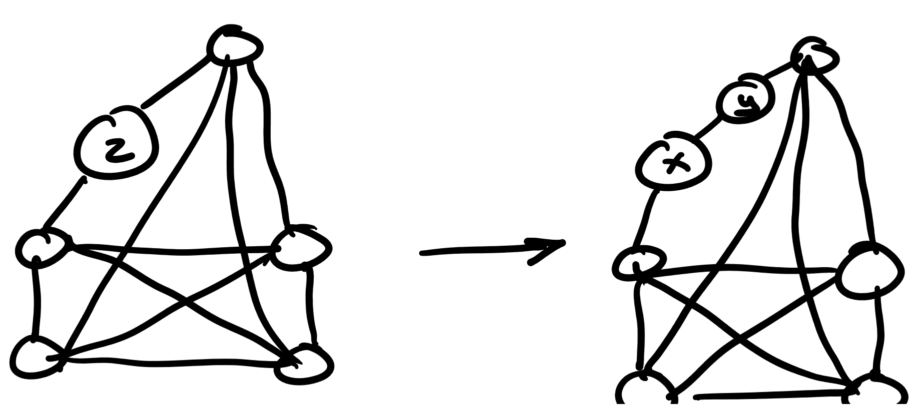


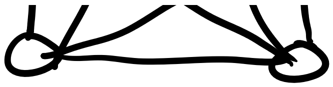
Case 1: $z \notin H$

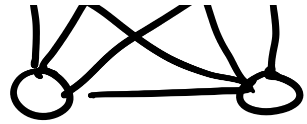
→ trivially holds

Case 2: $d_H(z) < 3$ ← degree of z within induced H

→ z is along a subdivided edge







Case 3: $d_H(z) \geq 3$

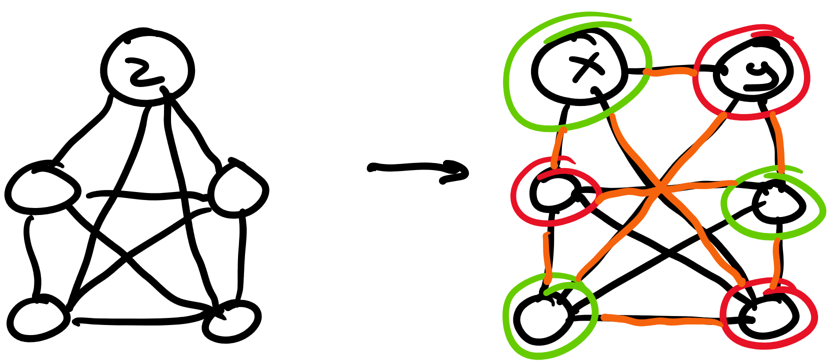
$d_H(x) \leq 2$ and $d(y) \leq 2$

→ same thing, e is along a subdivided edge

Case 4: $d_H(z) \geq 3$

$d_H(x) \geq 3$ and $d_H(y) \geq 3$

$K_5 \to K_{3,3}$ is the only way

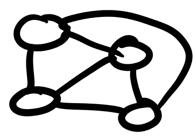


SO: Bring it on home via induction

7. If G is 3-connected with no K.S., then G has an embedding on the plane

on the plane

Induction on $|V(G)|$

Basis $\Rightarrow K_4$  is planar

Consider our $P(n)$ case

$\rightarrow \exists e$ s.t. $G \cdot e$ is 3-connected (5)

Note: $G \cdot e$ has no K.S. (6)

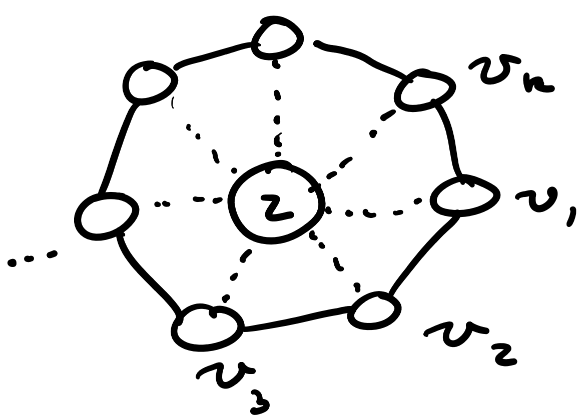
$\rightarrow P(k) = P(n) \cdot e$

I.H. on $P(k)$ gives us an embedding

Bring it back to $P(n)$

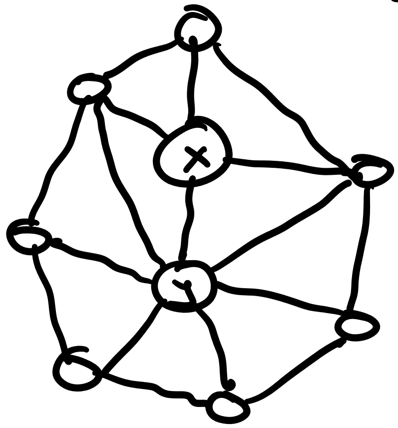
$\rightarrow$ consider $z \leftarrow (x,y) = e$

Note: all $N(z)$ can form a face that contains



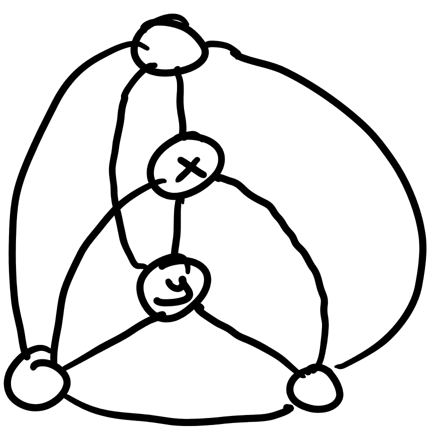
- order all $N(z)$ as $v_1 v_2 \ldots v_k$ around z
- consider embedding when $z \rightarrow (x,y)$

Case 1: $N(x)$ is some exclusive subset of $v_i \dots v_j$



$\rightarrow$ trivial to construct an embedding on $P(n)$

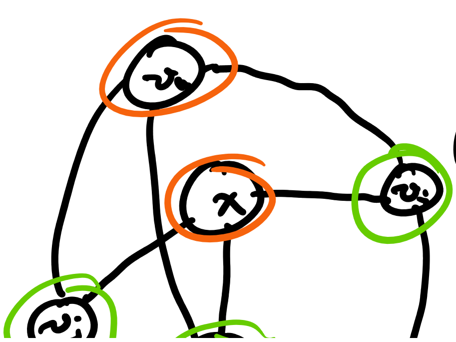
Case 2: $|N(x) \cap N(y)| \geq 3$



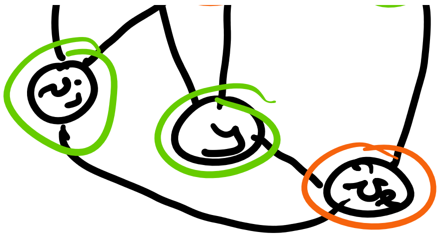
$\rightarrow$ we have K_5 K.S.

Case 3: $N(x)$ alternates with $N(y)$ s.t. $v_i v_j \in N(x)$
$v_\ell v_m \in N(y)$

$v_i < v_\ell < v_j < v_m$



$\rightarrow$ we have a $K_{3,3}$ K.S.



(4) + (7) = Kuratowski's theorem

any minimal counter-example must be 3-connected

any 3-connected counter-example doesn't exist

QED ☑