20.1 Conditions for Planarity

So far, we've come up with a few conditions to determine whether or not a graph G is planar. We've observed that K_5 does not have a planar embedding. Similarly, neither does $K_{3,3}$. So obviously, any graph that has K_5 or $K_{3,3}$ as a subgraph is not planar. Additionally, we've used Euler's formula to show how all planar graphs have $m \leq 3n - 6$, where $m = |E(G)|$ and $n = |V(G)|$. When G is triangle-free, then $m \leq 2n - 4$.

So for now, we know that a graph is not planar if:

1. It has K_5 as a subgraph
2. It has $K_{3,3}$ as a subgraph
3. $m > 3n - 6$
4. $m > 2n - 4$ if G is triangle-free

Note that in terms of determining if G is planar, we've only shown that these conditions are necessary but not sufficient. E.g., a graph with $m \leq 3n - 6$ is not necessary planar – think of $K_5 + v$, where v is a single additional vertex attached by a single edge to some $u \in K_5$ ($m = 11$, $n = 6 \rightarrow 11 < 12$, but we know a graph with a K_5 subgraph can't be planar).

Let's explore further conditions. Recall a subdivision, which is created by replacing a single edge with a path. Note that subdividing an edge does not affect planarity, since an embedding of a subdivided edge can be used to create an embedding of the original graph and vise-versa. Therefore, we can see that a planar graph cannot contain a subgraph that is a subdivision of K_5 or $K_{3,3}$. These subgraphs, subdivisions of K_5 and $K_{3,3}$, are called **Kuratowski subgraphs**.

20.2 Kuratowski's Theorem

Kuratowski's Theorem is the much stronger statement that a graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$. To prove Kuratowski's Theorem, we'll show the following:

1. For every face F_i of an embedding of G, it's possible to draw a new embedding of G with F_i as the outer face.
2. A **minimal nonplanar graph** is a nonplanar graph such that any subgraph is planar. Every minimal nonplanar graph is 2-connected.

3. An S-**lobe** of G is an induced subgraph consisting of a vertex set S as well as the vertices of some component of $G - S$. If $S = \{x, y\}$ is a separating set of nonplanar graph G, then adding the edge $e = (x, y)$ to some S-lobe of G yields a nonplanar graph.

4. If nonplanar graph G has the fewest edges among all nonplanar graphs without Kuratowski subgraphs, then G is 3-connected.

5. Every 3-connected graph G with at least five vertices has an edge e such that $G \cdot e$ is 3-connected.

6. If $G \cdot e$ has a Kuratowski subgraph, then G has a Kuratowski subgraph.

7. A convex embedding of a graph is an embedding where each face is a convex polygon. If G is 3-connected with no subdivision of K_5 or $K_{3,3}$, then G has a convex embedding on the plane with no 3 vertices in a line.

If G has a convex embedding, then obviously it must be planar. Therefore, any graph that contains a Kuratowski subgraph is nonplanar.