Lecture 21 - 4-color Theorem

Wednesday, April 3, 2024 5:25 PM

WP 11: show all planar graphs are a subgraph of some triangulation

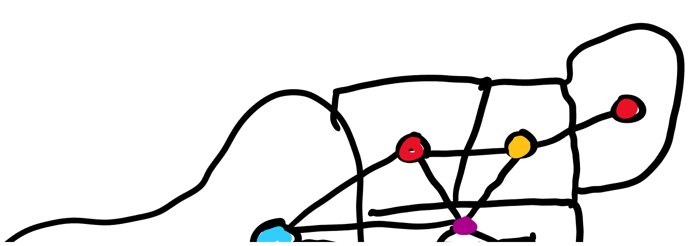
↳ maximal planar graphs → triangulations

Why we care:

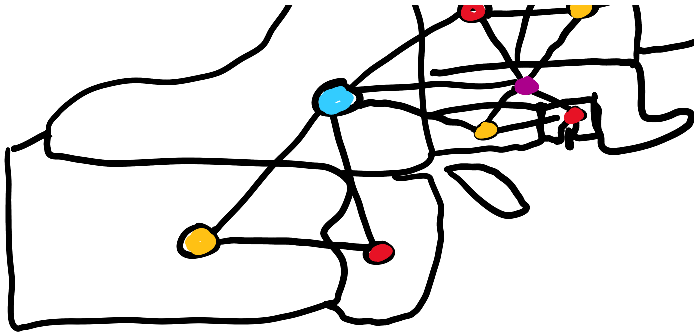
- Kuratowski Theorem
- 4/5 - color theorems centrally use this

↳ if we prove for triangulations, we prove for all graphs

Motivation: Map Coloring



How many colors are



colors are needed s.t. no neighboring regions have the same color?

→ Equivalent to coloring a planar graph

Big dogs:

Havel-Hakimi

Kuratowski

4-color theorem

Q: How many colors are needed for a planar graph?

sub Q: can we bound $\chi(G) \leq 5$ when G is planar

5-color theorem: yes we can

Note: any counter-example must have some v s.t. $d(v) \le 5$

↳ consider $m \le 3n-6$

and $\sum d(v) = 2m$

if all $d(v) = 6$

$\rightarrow 2m = \underbrace{6n}$

$m \le 3n-6$

$2m \le 6n-12$

$6n \le 6n-12$

$0 \le -12$

$\Rightarrow \exists v : d(v) \le 5$ in all planar graphs

5-color Theorem: $\chi(G) \le 5$

if G is planar triangulation

triangulation

To show: Induction on $|V(G)|$

Basis $P(\leq 5) \to$ trivial to color

$P(n)$ is a planar G
where $n = |V(G)|$

$P(k) = P(n) - v$
where v has $d(v) \leq 5$

From Kuratowski $\to$ deleting a
vertex can't create a K.S.
$\to P(k)$ is planar

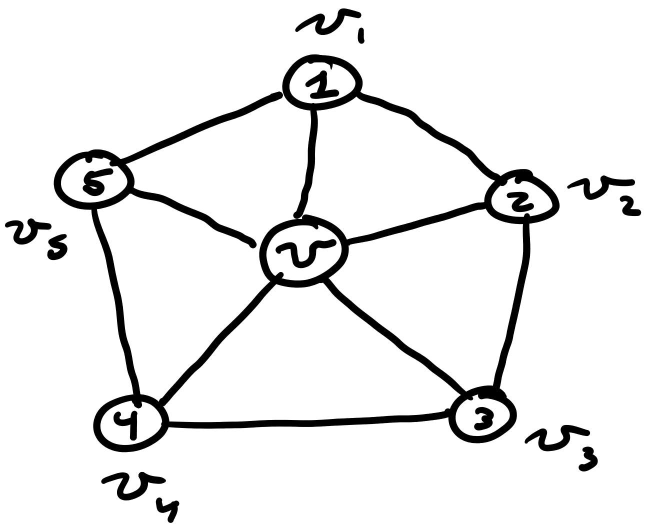
I.H. on $P(k)$ gives a 5-coloring

Bring it on back to $P(n)$

Case 1: $d(v) \leq 4 \to$ trivial

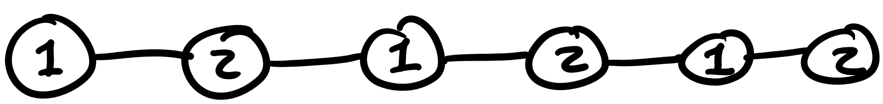
Case 2: $d(v) = 5$ and fewer than
5 colors in $N(v) \to$ trivial

Case 3: $d(v) = 5$ and 5 colors exist in $N(v)$

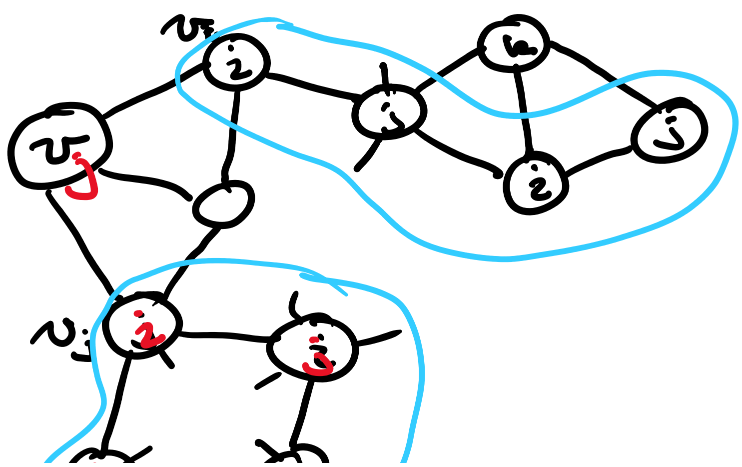


show: this configuration can be reduced or colored with 5-colors

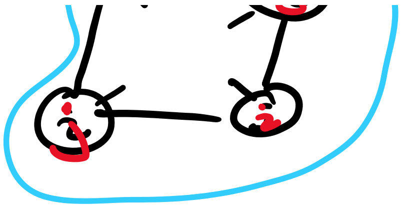
To do so: Kempe Chains aka color-alternating paths



Consider all possible Kempe chains around v for all possible i, j color pairs



* if for a given i,j pair of colors, no i,j-alternating path exists connecting

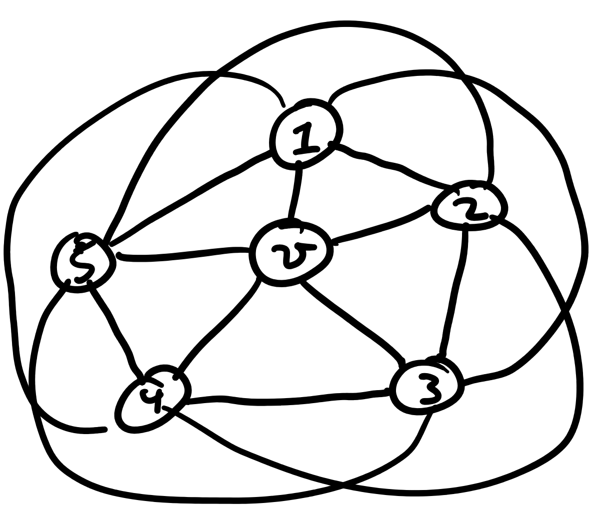


exists connecting $v_i \in N(v)$ and $v_j \in N(v)$

→ we can swap i, j colors on a maximal induced connected subgraph with colors i, j connected to either v_i or v_j

↳ this reduces the # colors in $N(v)$ by 1, we apply color i/j on v

Q: will there be an i,j-alternating path for all i,j pairs?



→ we see a K.S.

↳ so at least one i,j pair does not have a connecting Kempe chain $\square$

What about 4-colors?

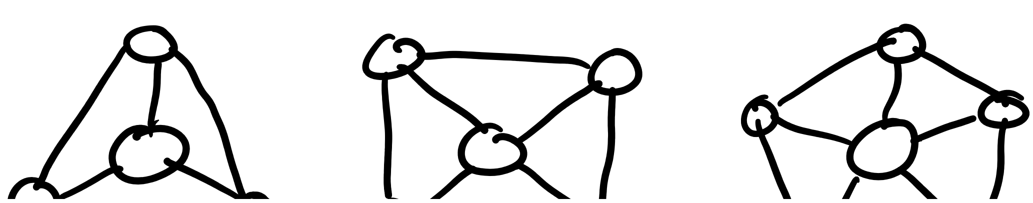
Let's give it a go with the above approach

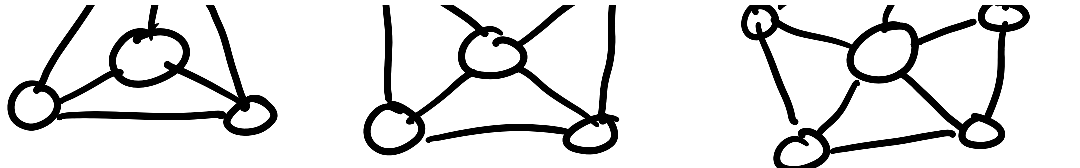
→ we're looking to find some minimal unavoidable configuration that a counter-example must contain

If we can all such possible configurations are reducible to color v with $1 \ldots 4$

$\Rightarrow$ all planar G are 4-colorable

Our 5-color theorem configurations





$d(v)=3$ $d(v)=4$ $d(v)=5$

Let's try the same approach to prove the 4-color theorem

Basis $P(\le 4) \to$ trivial

$P(n)$: planar G with $v : d(v) \le 5$

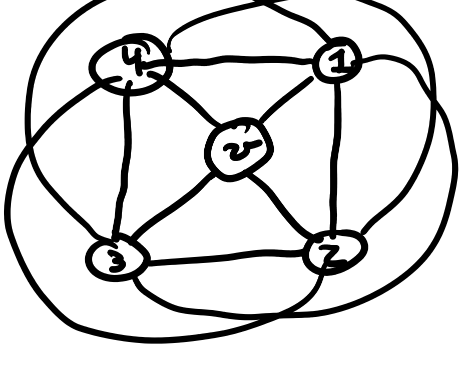
$P(k) = P(n) - v$

I.H.: $P(k)$ is 4-colorable

Bring it back to $P(n)$

Case 1: $d(v) = 3 \to$ trivial

Case 2: $d(v) = 4$



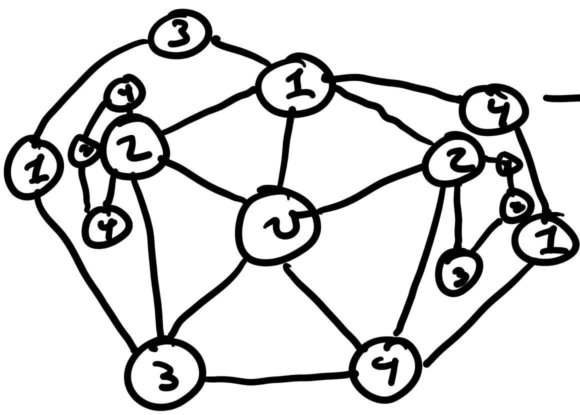
$\to$ at least one i,j pair must no have a connecting i,j-alternating path (otherwise K_5 K.S.)

(otherwise K_5 K.S.)

$\rightarrow$ we can reduce and give v color i w!o.g.

Case 3: $d(v) = 5$

Note: exactly two vertices have the same color in $N(v)$



$\rightarrow$ consider paths from $1 \rightarrow 3$ and $1 \rightarrow 4$

We can eliminate color 1 if these paths don't exist

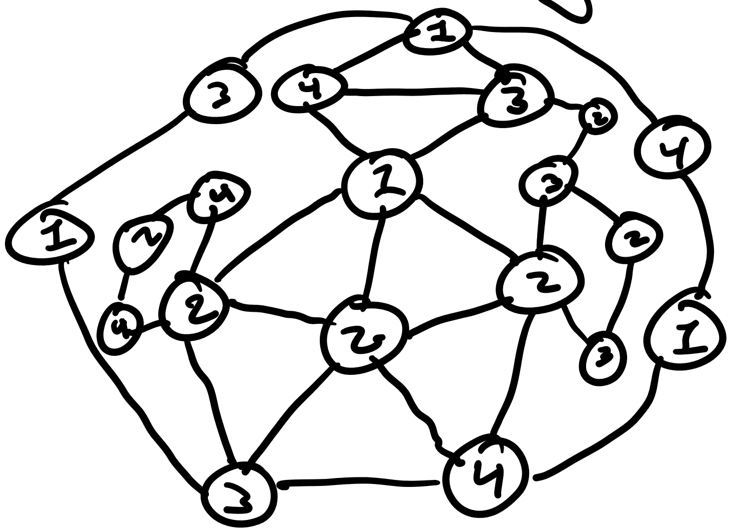
Now consider vertices w/ color 2

As we can't have a 2,3-alternating path $\rightarrow$ we can swap colors to get rid of one color 2 in $N(v)$

$\rightarrow$ do the same for 2, 4 to get rid of second color 2 in $N(v)$

$\Rightarrow$ give v color 2 and we're done $\square$

Actually no



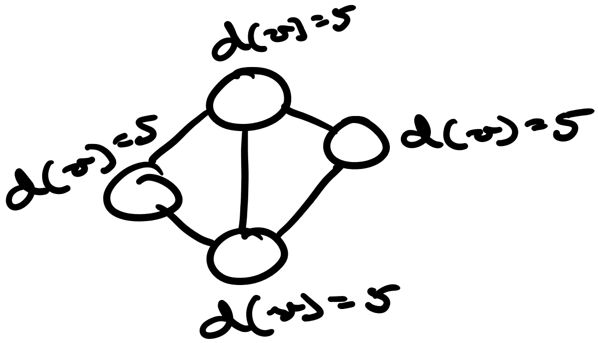
NOPE

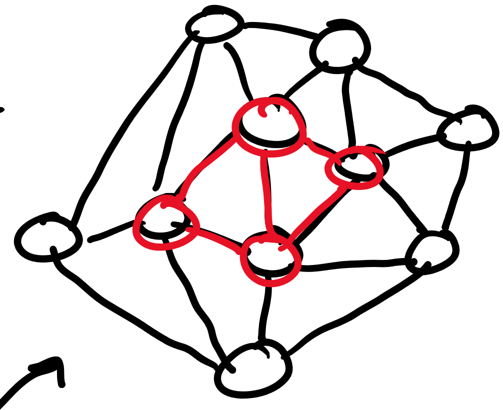
↳ As this configuration isn't reducible, we need to consider larger configurations

Up until now, we were considering configurations w.r.t. a single v

↳ we need to consider multiple v

E.g. Birkhoff diamond





How can we reduce this configuration?

Hint: Show all proper 4-colorings of the surrounding cycle can result in a 4-coloring with the diamond

The actual 4-color theorem:

- Generate all possible configurations
- Show all of them are reducible

reducible

→ then we're done

How many configurations:

O.G.: 1800

Now: 600

Q: How can generate all configurations?

$\sum d(v) = 6n - 12$

↪ $12 = \sum (6 - d(v))$

"charge" of v

As +12 as a sum is fixed

↪ at least some v have a positive charge

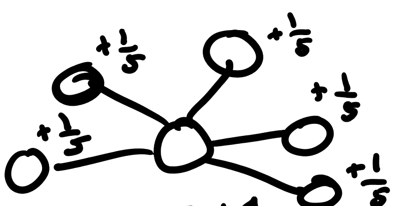
We can generate configurations by considering how charge is distributed

→ <u>discharge rules</u>

Consider some charge in a triangulation with $\delta(G) \geq 5$

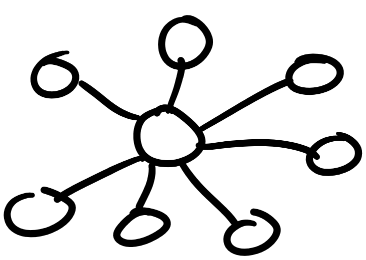
Our first discharge rule:

→ take and distribute some positive charge on v equally to all $N(v)$



$5 + 1$

← if one of these $u \in N(v)$ has positive charge, it must have started with $d(u) = 5$ or 6



$d(v) = 7$

→ if this ends up with a positive charge, it must have at least

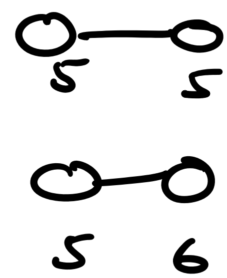
$d(v)=7$ must have at least

$\{6$ $d(u)=5$ verts in $N(v)$

in $N(v)$, at least 2 degree-5 u are connected

(same with degree-6 v)

$\Rightarrow$ this gives us



for induction

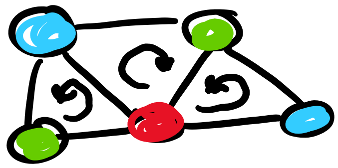
Repeating the above for various discharge rules gives us all configurations

reduce them and we're done $\square$

WP 11: First part $\rightarrow$ trivial

(consider a construction)

Part 2: consider how the faces are colored



(consider walks on the dual)