21.1 Coloring of Planar Graphs

One of the more well-studied problems related to coloring and graph planarity is the question of how many colors are required in order to color a map such that any two neighboring regions have different colors. As a map can be represented as a planar graph, where each region is given by a vertex and neighbor relations are given by edges between vertices, the problem is equivalent to asking what is the maximum chromatic number of a planar graph.

21.2 Five-color Theorem

Kempe Chains are color-alternating paths, or paths that alternate between two colors in a properly colored graphs. We can use Euler's formula, the degree sum formula, and Kempe Chains to show that every planar graph is 5-colorable. This is the **Five Color Theorem**. So we know that the chromatic number of all planar graphs is bounded by $\chi(G) \leq 5$. Is this the lowest bound we can establish? We'll talk about lowering this bound next.

21.3 4-Color Theorem

Let's try and reduce our bound to four colors. An approach for the proof involves us seeking some minimal counterexamples, which is an unavoidable set of graphs that can't be present in a four-colorable planar graph. We consider triangulations, since every simple planar graph can be contained in a triangulation.

A **configuration** in a planar triangulation is a separating cycle C with some portions of G inside of C. A configuration is **unavoidable** if a minimal counterexample must contain it. A configuration is **reducible** if a planar graph containing it cannot be a minimal counterexample.

Since the minimum degree of a planar graph is less than or equal to five and in a triangulation the minimum degree is greater than or equal to three, we immediately have three unavoidable configurations. For these, C is length three, four and five each with single vertex in the middle attached to each vertex in C.

We can try and prove that planar graphs are four-colorable using these configurations. However, the configuration using a cycle of length five isn't shown as reducible, so larger configurations must be considered.

The **Four Color Theorem** states that every planar graph is four-colorable. The proof originally involved demonstrating 1,834 unavoidable reducible configurations. Over the years, this was eventually reduced to "only" 633 unavoidable reducible configurations. Obviously, we aren't going to work through each configuration. It was the first major theorem utilizing computation as part of its proof, using it to show all configurations were reducible. No smallest counterexamples exist because they must contain, yet do not contain, one of these configurations. This contradiction means there are no counterexamples at all and that the theorem is therefore true.