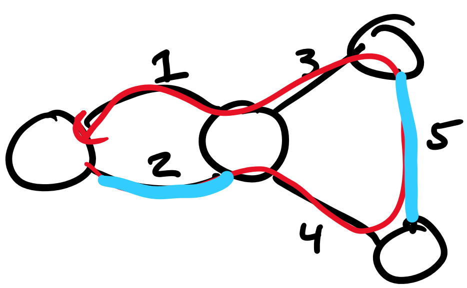
Lecture 22 - Line Graphs, Edge Coloring

Monday, April 8, 2024 11:15 AM

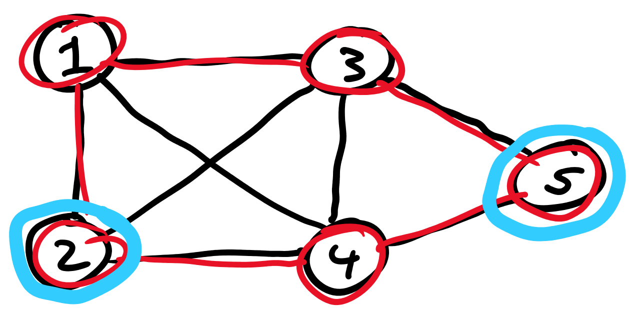
Line Graphs

The line graph of $G \to L(G)$

defined as:
- $E(G) \to V(L(G))$
- $E(L(G))$ is defined via shared endpoints of edges in G



G



$L(G)$

Note: each $v \in V(G) \to K_{d(v)}$ in $L(G)$

Note x2: the equivalence of $E(G) \to V(L(G))$ is relevant to several problem we've discussed to several problem we've discussed

1. Euler Tour on G

$\iff$

spanning cycle on $L(G)$

2. Matching on G

$\iff$

independent set on $L(G)$

3. cut edge on G

$\iff$

cut vertex on $L(G)$

4. edge coloring on G

$\iff$

vertex coloring on $L(G)$

Edge Coloring

Assign labels to all $e \in E(G)$

proper: no two edges that share an endpoint can have the same label (color)

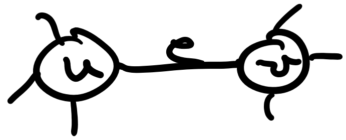
Edge Chromatic Number $\chi'(G)$

→ the minimum # of colors to properly edge color G

Let's Get Boundin'

$\chi'(G) \geq \Delta(G)$ as the largest degree vertex requires separate colors for all incident edges

Consider Greedy Coloring



$$d(u) = \Delta(G) = d(v)$$

$$\chi'(G) \le 2\Delta(G) - 1$$

via worst-case greedy coloring

$\chi'(G) = \Delta(G)$ if G is bipartite

Note: k-regular graphs have a perfect match

↳ all bipartite graphs are a subgraph of a k-regular graph

↳ color our P.M., remove it, color a new P.M., remove, repeat k times to get a k-edge-coloring on G

Q: can we tighten our lower bound in general?

A: yup

=> Show $\chi'(G) = \Delta(G)$

OR $\chi'(G) = \Delta(G) + 1$

(for simple G)

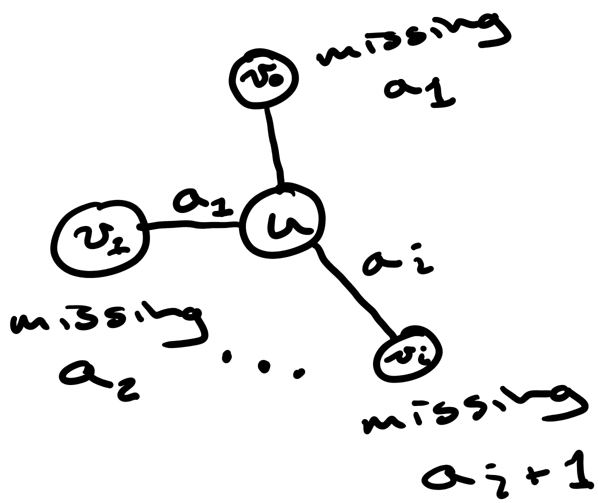
PROOF BY ALGO.

Consider f as some $\Delta(G)+1$ edge-coloring of some subgraph $H \subseteq G \rightarrow$ extend to all of G

Consider $u \in V(G)$ and edge $(u, v_0) \in E(G)$ with no color

In $N(u)$, there are some colors

In $N(u)$, there are some colors missing $\rightarrow a_0$ is one such color



- Consider u's neighbors
- Label $N(u)$ s.t. $a_{i+1} \rightarrow$ color missing at vertex $v_i \in N(u)$

If color a_0 is not in $N(v_0)$ $\rightarrow$ color (u, v_0) with a_0

If color a_1 is not in $N(v_0)$ and a_1 is not in $N(u)$ $\rightarrow$ color (u, v_0) with a_1

If a_2 is missing at v_1, there must exist (u, v_2) with color a_2, otherwise we can just color (u, v_0) with a_1

Generally: if a_i is missing, we can use a_i on (u, v_{i-1}) and "shift" our colors down to eventually color (u, v_0) with a_1

→ either a missing color repeats or this procedure is possible since we have at most $\Delta(G)+1$ edge colors

v_ℓ is the first vertex with a missing color on $a_1 \dots a_\ell$

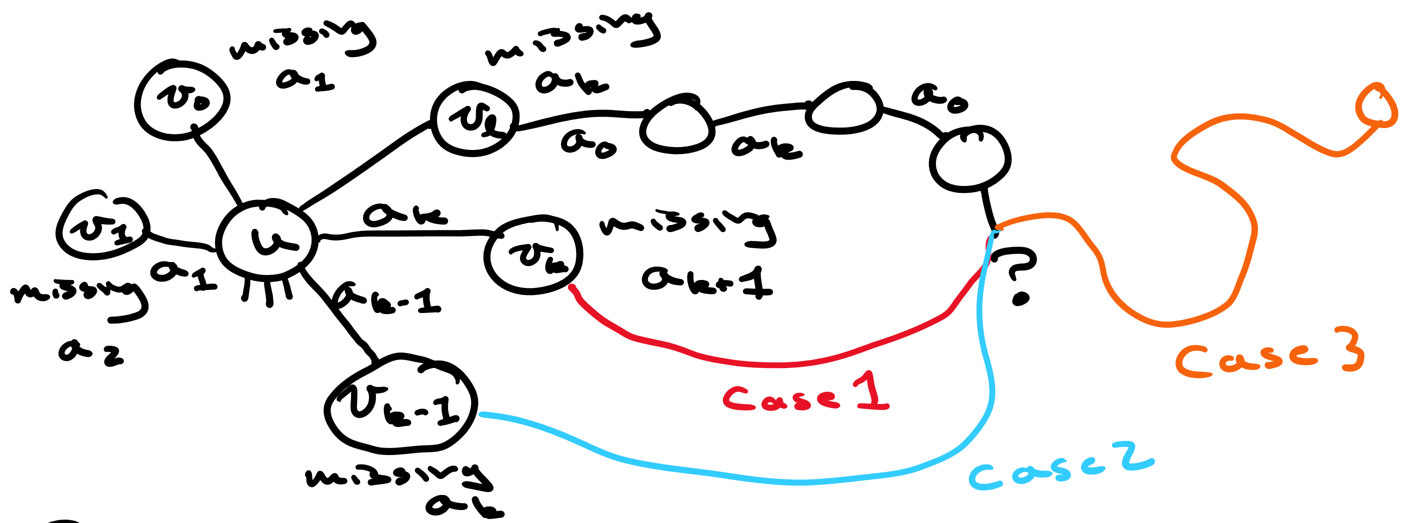
→ we'll call it a_k

Note: also missing at v_{k-1} and it is on edge (v_k, u)

Note x2: a_0 also appears on v_ℓ otherwise we could color (u, v_ℓ) with a_0 and shift colors down

Exxxxtreeemal Argument

Consider P as a maximal $a_0 a_k$ edge alternating path from v_ℓ



Case 1: P reaches v_k

→ shift colors down from v_{k-1} and swap colors on P, edge (u, v_ℓ) gets color a_k

Case 2: P reaches v_{k-1}

→ shift down from v_{k-1}, put a_0 on (u, v_{k-2}) and swap colors on P colors on P

Case 3: P reaches elsewhere

→ shift colors down from v_e, put a_0 on (u, v_e) and swap colors on P

⇒ no matter what, all graphs can be colored in at most $\Delta(G)+1$ colors

$$\boxed{\Delta(G) \le \chi'(G) \le \Delta(G)+1} \quad \square$$

As we discussed

if $\exists H$ s.t. $L(H) = G$

→ we can get a max. ind. set on G in polynomial time

→ we can get an optimal

→ we can get an optimal vertex coloring in poly. time

Our Q: given some G, does there exist some H where G is the line graph of H?

A: sometimes

O: if such an H existed obviously for all G → P = NP

Q2: Can we characterize G where such an H exists?

$L(H) = G$

From earlier

→ a line graph can decompose into maximal cliques with each vertex in at most 2 vertex in at most 2

→ this must hold for any G
where $L(H) = G$

This gives us a necessary

condition for G

→ is it sufficient

For simple G, $\exists H$ s.t. $L(H) = G$
iff G decomposes into
maximal cliques where vertex
is in at most 2

($\Rightarrow$) Note: every vertex in H
becomes a clique in G
And: every edge in H is
attached to at most 2
maximal cliques as a
vertex in G

cliques as a vertex in G

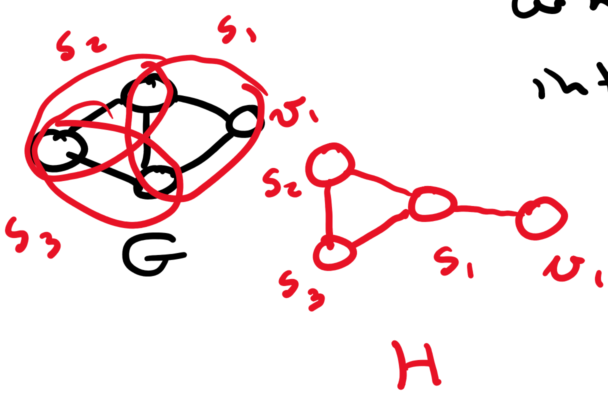
(⇐) define: $S_1\, S_2 \ldots S_k$ as vertex sets of maximal cliques in a decomposition of G

Construct H:

$v_1 v_2 \ldots v_e$ are vertices in only 1 of S_i

$V(H) = \{$ one vertex for each in $S_1 \ldots S_k\ v_1 \ldots v_e \}$

$E(H) = \{$ for all $(v_i S_j)$ and $(S_n S_m)$ where these pairwise intersect $\}$



→ each $v \in V(H)$ is in at most two sets S_i with no vertices two sets S_i with no vertices in the same two sets

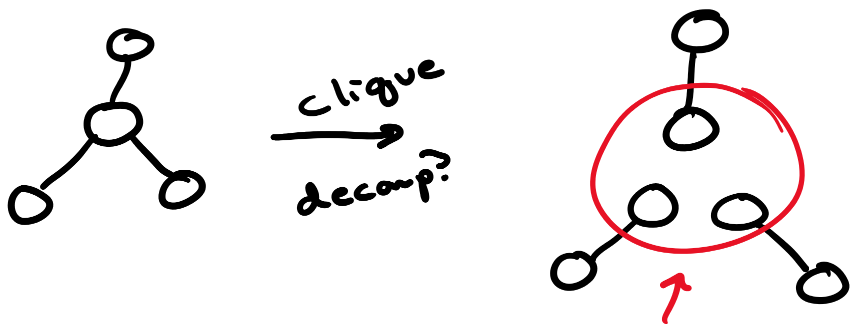
$\Rightarrow$ together, this implies the existence of our H s.t. $L(H) = G$ $\square$

Big Q:

Is there a simpler way to characterize G?

Big A: yes

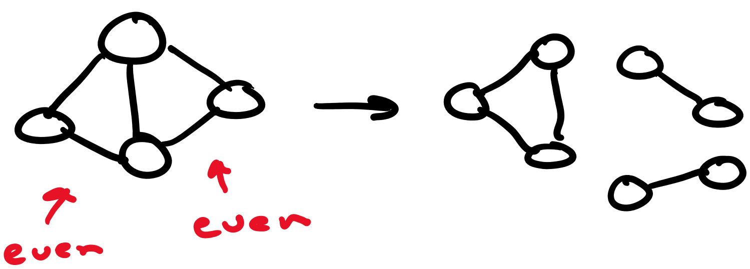
Consider an induced claw



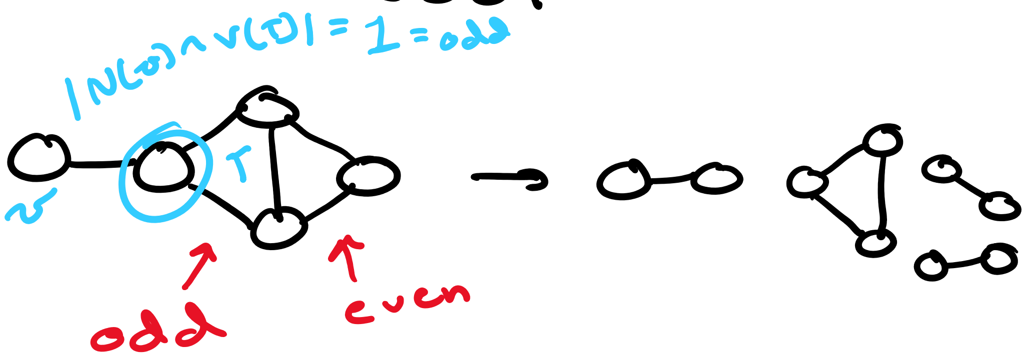
center vertex is in at least 3 max cliques

$\Rightarrow$ no such G can have an induced claw as a subgraph

Consider a double triangle



what aboot:

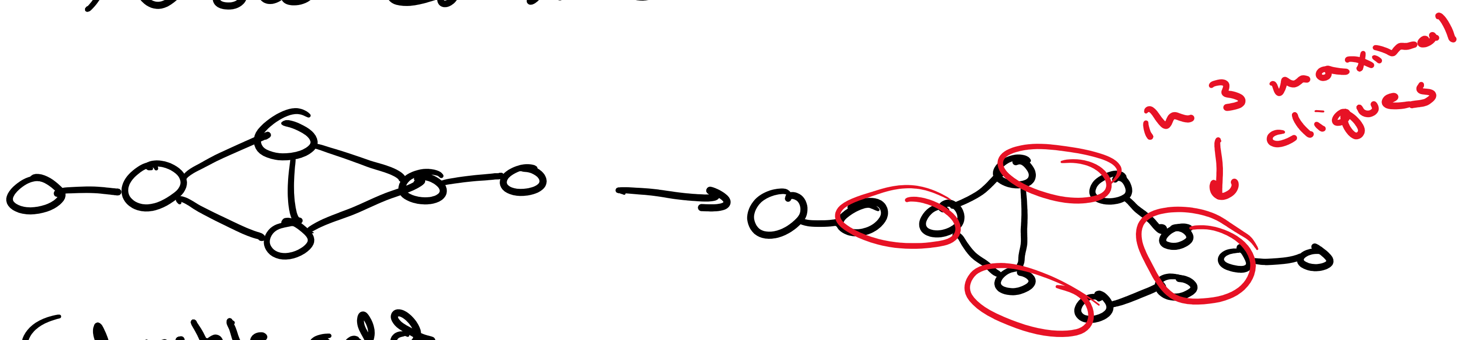


odd triangle T: $\exists v \in V(G)$, $T \subseteq G$ s.t. $|N(v) \cap V(T)| = \text{odd}$

even triangle T: $\forall v \in V(G)$, $T \subseteq G$

s.t. $|N(v) \cap V(T)| = \text{even}$

Now consider



(double odd triangle)

$\Rightarrow$ no such G can have a double odd triangle as an induced subgraph

Necessary conditions for G:

G has no claws

G has no D.O.T.s

double odd triangles

Q: sufficient?

Next class