Lecture 23 - Hamiltonian Graphs

Sunday, April 14, 2024 12:32 PM

To prove: $\exists H$ s.t. $G = L(H)$

iff

G has no claws or double odd triangles (DOTS)

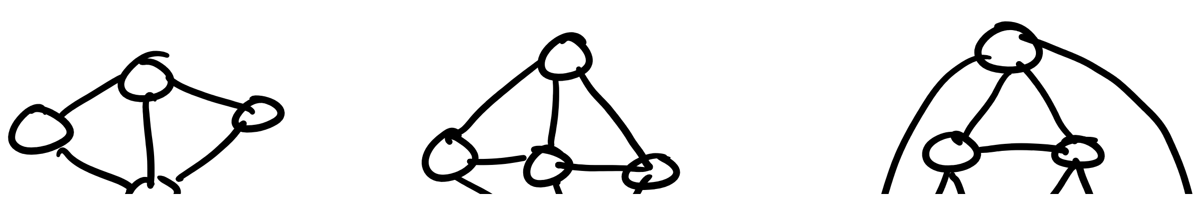
($\Rightarrow$) Contrapositive

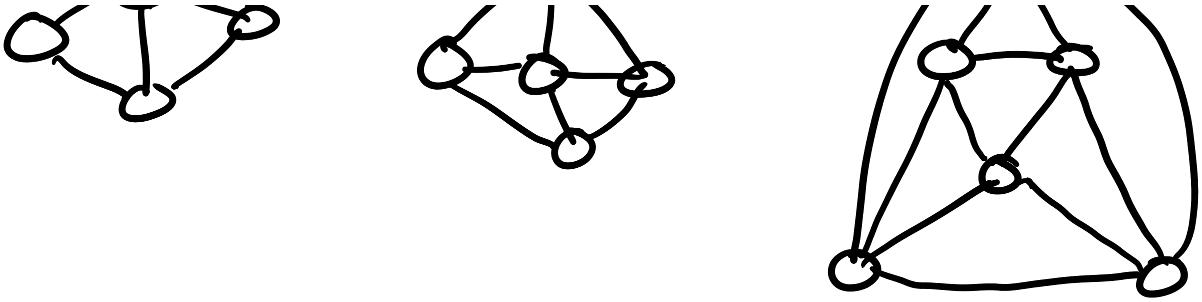
G has claws / DOTs $\Rightarrow$ $\exists$ no such H

$\Rightarrow$ we already proved this last class

($\Leftarrow$) First: consider double even triangles

$\rightarrow$ only 3 such graphs exist for simple connected graph





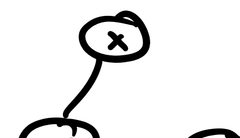
$\Rightarrow$ we only need consider double triangles with one odd and one even triangle

Consider a maximal clique decomposition, with one caveat:

$S_1 S_2 \ldots S_k$ are maximal cliques <u>except</u> for even triangles that aren't shared with an odd triangle

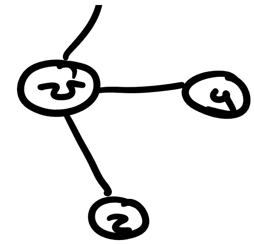
Q: $\forall v \in V(G)$, is v in at most 2 S_i, S_j?

Consider $v \in S_i\ S_j\ S_k$

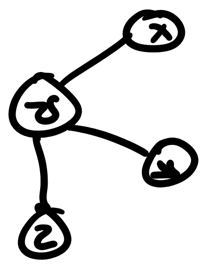


Consider $v \in S_i \, S_j \, S_k$

$\{x, y, z\} \in N(v)$

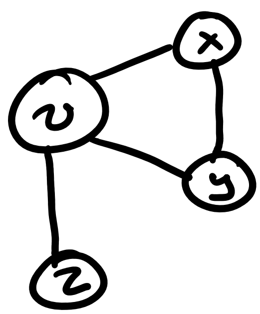


Case 1: no edges $(x,y), (y,z), (x,z)$



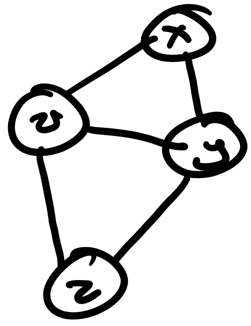
→ this is a claw, so this configuration does not exist

Case 2: edge (x,y) exists

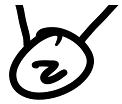


→ an odd triangle, but odd triangles should be in a single S_i given our decomposition

Case 3: edges (x,y) and (y,z)

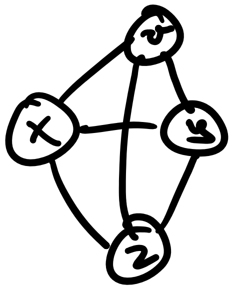


→ we have two even triangles, regardless of choice of v



of choice of v

Case 4: edges (x,y) (y,z) (x,z) exist



$\rightsquigarrow$ We have K_4, which should be in one S_i

$\Rightarrow$ Taken together, along with the assumed decomposition, v can be in at most two subgraphs

$\Rightarrow$ $\exists H$ s.t. $G = L(H)$ $\square$

All together

Our characterization of G

- G has no claws
- G has no DOTs

G has no OOTs

set of

Forbidden

subgraphs

Q: What are Hamiltonian graphs?

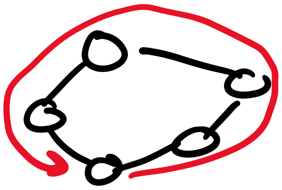
A: A graph with a spanning cycle

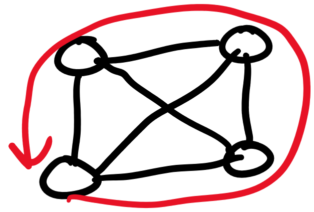
Hamiltonian cycles

<=>

Spanning cycles

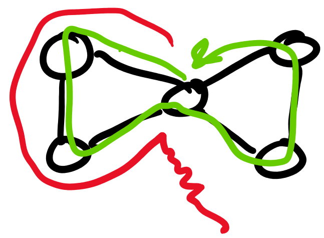
What graphs are Hamiltonian?



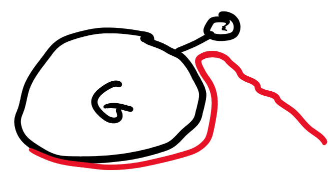


cycle graphs cliques

Eulerian?



Outerplanar?



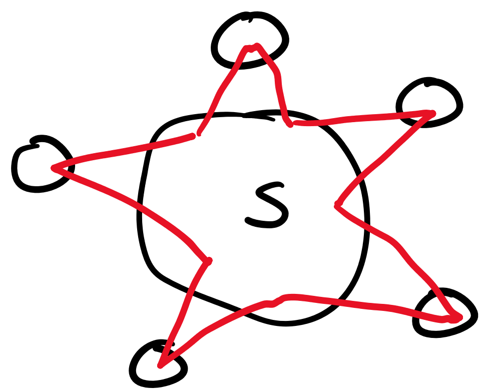
Necessary conditions
for Hamiltonian G

- G must be connected
- G must be 2-connected
  → spanning cycle can't contain a cut vertex

- If G is bipartite then $|X| = |Y|$
  → a cycle traverses between X and Y an equal number of times

- If $c(G)$ = # of components of G

  then $c(G-S) \le |S| \;\; \forall S \subseteq V(G)$



→ For each comp, we need a unique vertex in S to traverse to the next comp

Q: But what about sufficient conditions?

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sufficient conditions for Hamiltonian G

If $|V(G)| \ge 3$ and $\delta(G) \ge \left\lceil \frac{|V(G)|}{2} \right\rceil$

If $|N(v)| = 3$ and $\delta(G)$ $\left\lceil \frac{}{2} \right\rceil$

Consider maximal non-Hamiltonian G'

$\rightarrow$ $G' + e$ = Hamiltonian

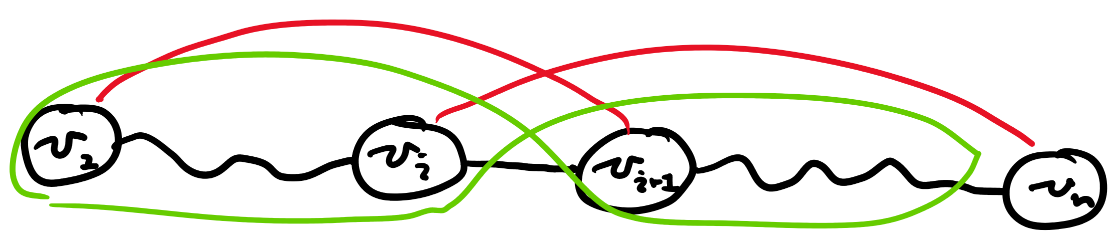
$\rightarrow$ G' has a <u>spanning path</u>

$\hookrightarrow$ Hamiltonian path

Consider this path in some order $P = \{ v_1 v_2 \dots v_n \}$

If along this path $\exists\, v_i v_{i+1}$ s.t. $v_i \in N(v_n)$, $v_{i+1} \in N(v_1)$

$\rightarrow$ we can construct a spanning cycle



define $S = \{ i : (v_1 v_{i+1}) \}$

define $T = \{ i : (v_n v_i) \}$

define $T = \{ i : (v_n, v_i) \}$

show $|S \cap T| \geq 1$

↪ we have the above and therefor a cycle

$$|S \cup T| + |S \cap T| = |S| + |T|$$

$$|S| + |T| = d(v_1) + d(v_n) \geq |V(G)|$$

$$|S \cup T| + |S \cap T| \geq |V(G)|$$

$|S \cup T| < |V(G)|$ as we assume no (v_1, v_n) edge

$$|S \cap T| \geq 1$$

=> we have a spanning cycle

Applying this logic to all possible pairs to get a spanning cycle in the spanning cycle in the

general case $\square$

If $\forall u, v \in V(G)$ $(u, v) \notin E(G)$

$$d(u) + d(v) \geq |V(G)|$$

G is Hamiltonian

iff

$G + (u, v)$ is Hamiltonian

($\Rightarrow$) trivial, as adding an edge won't delete a spanning cycle

($\Leftarrow$) this follows from our prior proof since $|N(u) \cap N(v)| \geq 1$ $\square$

The above can be used

to define the <u>closure</u> of G

<u>Closure of G</u>: add (u, v) to $E(G)$

$\forall u, v \in V(G)$

constructs some G'

Constructs some G' $\begin{cases} \forall u, v \in V(G) \\ \text{s.t. } d(u)+d(v) \geq |V| \end{cases}$

To rephrase (sp?) the prior proof:

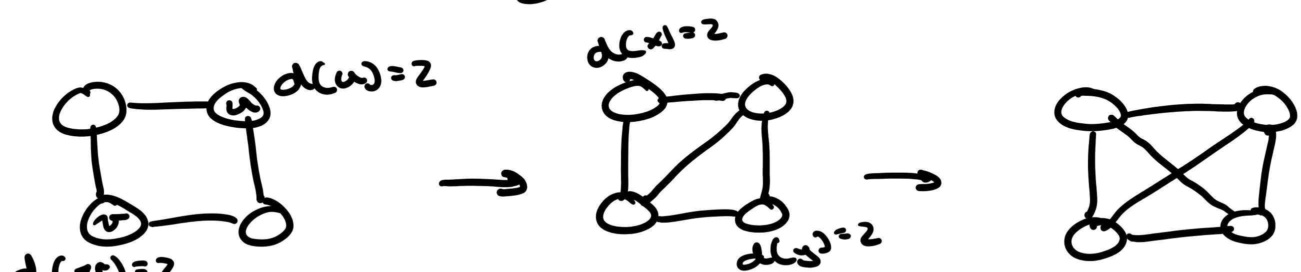
G is Hamiltonian

iff G's closure is

Hamiltonian

Let's LOOK at

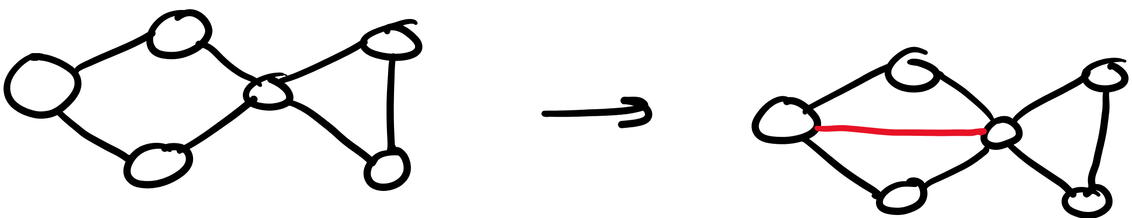
some closures



$d(u)+d(v) \geq |V(G)|$

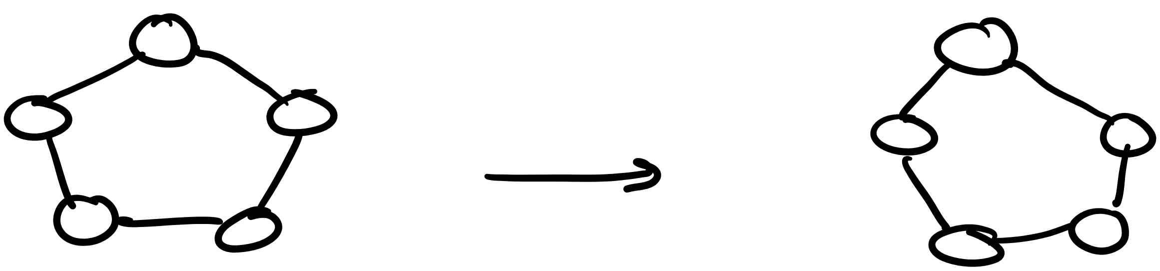
the closure of $C_4 = K_4$

G $\qquad\qquad$ K_6



G

closure of G is <u>NOT</u> Hamiltonian



C_5

closure of $C_5 = C_5$

NOT a clique

but still Hamiltonian

sufficient condition:

If G's closure is a clique, then G is Hamiltonian

Q: is the closure of G well-defined?

(we will always end up with the same closure irregardless of the order of edges that we add)

Consider

$e_1 e_2 \dots e_i$ and $f_1 f_2 \dots f_j$ are edges added to create the closure of $G \rightarrow G_e, G_f$

→ since e_1 can be added for G_e, it must be eventually added to G_f as some f_k

→ if any e_ℓ depends on e_1, there is equivalently some f_m that depends on f_k, so f_m <u>will</u> also be added to G_f

=> apply this logic to all

$\Rightarrow$ apply this logic to all $e_1 \ldots e_i$ we have an equivalent set of edges contained in $f_1 \ldots f_j$

$$\Rightarrow G_e \cong G_f \quad \square$$

Instead of explicitly constructing a closure of G

→ Look at the degree sequence of G to determine if the closure will be a clique

Chvátal's Condition

Consider G with degrees

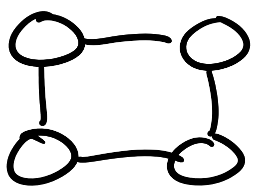
$$d_1 \le d_2 \le \ldots \le d_n$$

If $i < \frac{n}{2}$ implies $d_i > i$

or $d_{n-i} \geq n-i$

$\Rightarrow$ the closure of G is a clique

$\Rightarrow$ G is Hamiltonian $\square$



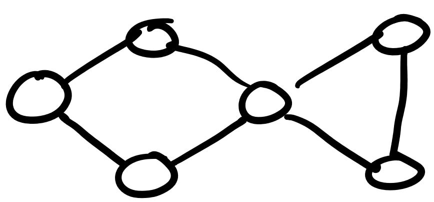
$S = 2\ 2\ 2\ 2$

$i = 1\ 2\ 3\ 4$

$n = 4$

$i = 1$

$d_1 = 2 > i = 1$ ✓



$S = 2\ 2\ 2\ 2\ 2\ 4$

$i = 1\ 2\ 3\ 4\ 5\ 6$

$n = 6$

$i = 1$

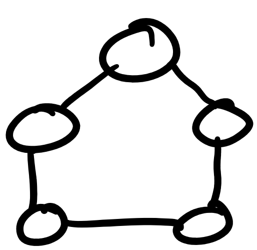
$d_1 = 2 > i = 1$ ✓

$i = 2$

$d_2 = 2 > i = 2$ ✗

$d_{n-2} = d_4 = 2 \geq 4$ ✗

→ condition doesn't hold



$S = 2\ 2\ 2\ 2\ 2$

$i = 1\ 2\ 3\ 4\ 5$

$n = 5$

$i = 1$

$d_1 = 2 > 1$ ✓

$i = 2$

$d_2 = 2 > 2$ ✗

$d_{n-2} = 2 \geq n-2 = 3$ ✗

→ condition doesn't hold

condition doesn't hold
but G is still
Hamiltonian

Next class:

We'll talk Ham. paths

and Erdös-Renyi
graphs