23.1 Hamiltonian Cycles

Recall that a **spanning subgraph** is subgraph in some graph that contains all vertices of that graph. A **spanning cycle** is a spanning subgraph that is a cycle. We're going to go a bit more in-depth with this concept. Not all graphs have a spanning cycle. Graphs with a spanning cycle are called **Hamiltonian Graphs**. Such a cycle is commonly referred to as a **Hamiltonian Cycle**. What properties must a Hamiltonian graph have? We're going to consider simple and connected (obviously) graphs here.

- A Hamiltonian graph must be biconnected.
- A Hamiltonian bipartite graph must have equal sized sets.
- If $c(H)$ is the number of components of a graph H, then Hamiltonian graph G must satisfy $c(G - S) \leq |S|$ for all possible $S \subseteq V(G), S \neq \emptyset$.

These give us several necessary conditions. But what about sufficient conditions?

23.2 Sufficient Conditions for Hamiltonian Graphs

For the following conditions and discussions, again consider all graphs as simple and connected.

If G has at least three vertices and $\delta(G) \geq \frac{n}{2}$, then G is Hamiltonian.

If $\forall u, v \in V(G) : (u, v) \notin E(G), d(v) + d(u) \geq n$, G is Hamiltonian if and only if $G + (u, v)$ is Hamiltonian.

A **closure** of a graph G, $C(G)$, is the graph with vertex set $V(G)$ obtained from G by iteratively adding edges joining pairs of nonadjacent vertices whose degrees sum to at least n, until no such pair remains. A graph G is Hamiltonian if and only if its closure $C(G)$ is Hamiltonian. The closure of G is also well-defined.

Consider graph G with vertex degrees $d_1 \leq \cdots \leq d_n$, where $n \geq 3$. If $i < \frac{n}{2}$ implies that $d_i > i$ or $d_{n-i} \geq n - i$, then G is Hamiltonian. This would further imply that the closure of G is K_n. This is **Chvátals's Condition**. So if we were to ever determine that the closure of a graph is a clique, then we'd know that the original graph is also Hamiltonian.

A **Hamiltonian path** is a spanning path. A join between two graphs G and H, $G \vee H$, is the graph created by adding edges between all vertices of G with all vertices of H. Or if $J = G \vee H$, $V(J) = V(G) \cup V(H)$ and $E(J) = E(G) \cup E(V) \cup \{\forall u \in V(G), \forall v \in V(H) : (u, v)\}$. A graph G has a Hamiltonian path if and only if $G \vee K_1$ has a Hamiltonian cycle.

We can deduce a similar condition as above for spanning paths. For simple graph G with degrees $d_1 \leq \cdots \leq d_n$, if $i < \frac{n+1}{2}$ implies $d_i \geq i$ or $d_{n+1-i} \geq n-i$, then G has a spanning path. Think of this as Chvátals's Condition applied to some graph that is joined with K_1. If the join has a spanning cycle, then the graph itself must have a spanning path.