11.1 k-Connected Graphs

We can now further extend a few of the concepts we discussed with restriction to 2-connected and 2-edge-connected to k-connected and k-edge-connected graphs. Given two vertices $x, y \in V(G)$, a set $S \subseteq V(G) - \{x, y\}$ is an x, y-**separator** if $G - S$ has no x, y-path. We define $\kappa(x, y)$ as the minimum cardinality over all possible x, y-separators and $\lambda(x, y)$ as the maximum cardinality over all possible sets of internally disjoint x, y-paths. Since any x, y-separator must contain an internal vertex of every internally disjoint x, y-path, we have $\kappa(x, y) \geq \lambda(x, y)$.

What follows is a generalization of Whitney's Theorem. **Menger's Theorem** states that for two vertices $x, y \in V(G)$ and $(x, y) \notin E(G)$ the minimum size of an x, y-separator equals the maximum number of pairwise internally disjoint x, y-paths; i.e, $\kappa(x, y) = \lambda(x, y)$. A graph is therefore k-**connected** if for all $x, y \in V(G)$, $\lambda(x, y) \geq k$.

We have similar concepts and terminology for k-edge-connectivity. Given two vertices $x, y \in V(G)$, a set $F \subseteq E(G)$ is an x, y-**disconnecting set** if $G - F$ has no x, y-path. We define $\kappa'(x, y)$ as the minimum cardinality over all possible x, y-disconnecting sets and $\lambda'(x, y)$ as the maximum cardinality over all possible sets of edge disjoint x, y-paths. A graph is k-**edge-connected** if for all $x, y \in V(G)$, $\lambda'(x, y) \geq k$. Likewise, $\kappa'(x, y) = \lambda'(x, y)$.

11.2 Network Flow

Consider a directed graph G where each edge $e \in E(G)$ has a given **capacity** $c(e)$. We also have a distinguished **source vertex** s and **sink vertex** t. Such a graph is called a **flow network**.

A **flow** $f(e)$ on a flow network G assigns a value to each $e \in E(G)$. For each $v \in V(G)$ we have $f^+(v)$ as the sum of flows from incoming edges on v and $f^-(v)$ as the sum of flows on outgoing edges. For non-source and non-sink vertices, a flow is **feasible** if is satisfies constraints $\forall e \in E(G) : 0 \leq f(e) \leq c(e)$ and $\forall v \in V(G), v \neq s, t : f^+(v) = f^-(v)$. The **value** $\text{val}(f)$ of a flow f is the net flow into the sink, $f^-(t) - f^+(t)$. A **maximum flow** is a feasible flow where $\text{val}(f)$ is maximum.

When f is a feasible flow in a network, a f-**augmenting path** is a source-to-sink path P where for each $e \in P$:

1. if P follows e in a forward direction, then $f(e) < c(e)$
2. if P follows e in a backward direction, then $f(e) > 0$

Define $\epsilon(e) = c(e) - f(e)$ when e is forward on P and $\epsilon(e) = f(e)$ when e is backward on P. The **tolerance** of P is $\min_{e \in E(P)} \epsilon(e)$.

If P is an f-augmenting path with tolerance z, then changing flow by $+z$ on forward edges in P and $-z$ on backward edges in P produces a new feasible flow $\text{val}(f') = \text{val}(f) + z$.

In a flow network, a **source-sink cut** $[S, T]$ consists of the edges between a **source set** S and **sink set** T, where S and T partition the nodes and $s \in S, t \in T$. The **capacity** of the cut $[S, T]$, $\text{cap}(S, T)$ is the total capacities of the edges of $[S, T]$, with the net flow from S to T equal to $\text{val}(f)$ and $\text{val}(f) \leq \text{cap}(S, T)$. Among all possible $[S, T]$ cuts, the one with the lowest $\text{cap}(S, T)$ gives us a bound on our maximum flow. The **Max-flow Min-cut Theorem** states the duality between the maximum flow and **minimum cut** problems; specifically, the maximum value of a feasible flow equals the minimum capacity of a source-sink cut.