# CSci 6974 and ECSE 6966
# Math. Tech. for Vision, Graphics and Robotics
# Lecture 1, January 19, 2006
# Points, Lines, Planes and Vectors

## Motivation

- There are several ways to motivate linear and matrix algebra:
  - Algebraic operations on points, lines and planes
  - Solving systems of linear equations
  - Solving differential equations
- I have chosen the first and will focus on geometric intuitions throughout our discussion.

## Points

- A point is a location in n-dimensional space.
- A point can be described by a tuple of scalars $(x_1, \ldots, x_n)$. This tuple is a **vector**.
- Euclidean $n$ space $\mathbb{R}^n$ is the set all of point location vectors together with a distance metric on the points.
- The distance between two points $\mathbf{x} = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ and $\mathbf{y} = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$ is

$$\sqrt{(x_1 - y_1)^2 + \ldots (x_n - y_n)^2} = \left[ \sum_{i=1}^{n} (x_i - y_i)^2 \right]^{1/2} \tag{1}$$

## Vectors

- Vectors are used to describe point locations and directions. The geometric intention of what is meant will be determined by the context.
- Vectors have an algebra, including a series of important properties. What follows is a summary.
- The length, magnitude or norm of a vector $\mathbf{x} = (x_1, \ldots, x_n)$ is

$$\|\mathbf{x}\| = (x_1^2 + \ldots + x_n^2)^{1/2} = \left[ \sum_{i=1}^{n} x_i^2 \right]^{1/2}. \tag{2}$$

  This may be proved by induction using the Pythagorean Theorem.

- When a vector $\mathbf{x}$ is multiplied by a scalar, $c$, the result is a vector with each of its components scaled individually.

$$c\mathbf{x} = (cx_1, \ldots, cx_n). \tag{3}$$

- The *sum* of two vectors, $\mathbf{x}$ and $\mathbf{y}$, is another vector formed by adding each of the respective components.

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n). \tag{4}$$

  We will look at the geometric interpretation of this.

- The *dot* or *scalar* product of two vectors, $\mathbf{x}$ and $\mathbf{y}$, is a scalar.

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \ldots + x_n y_n = \sum_{i=1}^{n} x_i y_i. \tag{5}$$

- In $\mathbb{R}^3$, the cross-product of two vectors is another vector:

$$\mathbf{x} \times \mathbf{y} = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1). \tag{6}$$

- Note that $\|\mathbf{x}\| = (\mathbf{x} \cdot \mathbf{x})^{1/2}$
- A vector $\mathbf{x}$ is a unit vector if $\|\mathbf{x}\| = 1$
- We use a special notation for a unit vector: $\hat{\mathbf{x}}$.
- A vector can be converted to a unit vector — just divide by the magnitude:

$$\hat{\mathbf{x}} = \frac{\mathbf{x}}{\|\mathbf{x}\|}. \tag{7}$$

- The angle between two vectors is found from the dot product:

$$\cos\theta = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\| \, \|\mathbf{y}\|}. \tag{8}$$

- We can decide if two vectors are perpendicular, parallel, or antiparallel by looking at the dot product between the vectors. These follow from the definition of the angle between two vectors.
- The direction between two points is the vector joining them, i.e. $\mathbf{x} - \mathbf{y}$. It is unique only up to a scale multiple.
- Vectors can be projected onto each other. In particular, if $\hat{\mathbf{y}}$ is a unit vector then

$$(\mathbf{x} \cdot \hat{\mathbf{y}})\hat{\mathbf{y}} \tag{9}$$

  is a vector in direction $\hat{\mathbf{y}}$.

- Given a vector $\mathbf{x}$ and a unit vector $\hat{\mathbf{y}}$, $\mathbf{x}$ can be decomposed into two vectors, one parallel to $\hat{\mathbf{y}}$ and one perpendicular to $\hat{\mathbf{y}}$
- Using all of the above, we can sketch a geometric argument showing the relationship between the cosine and the dot product (8).

## Lines

- You probably first learned the algebraic definition of a line in $\mathbb{R}^2$ using the "slope-intercept" form:

$$y = mx + b \tag{10}$$

  This is a rather limited definition, since it can not represent vertical lines.

- A line may be specified by a point, $\mathbf{x}_0$ and a direction $\mathbf{d}$. This gives the parametric form for the line.

$$\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{d}. \tag{11}$$

- A line in $\mathbb{R}^2$ may also be specified by a point $\mathbf{x}_0$ and a $\boldsymbol{\eta}$ normal to the line. In this case a point $\mathbf{x}$ is on the line if

$$(\mathbf{x} - \mathbf{x}_0) \cdot \boldsymbol{\eta} = 0. \tag{12}$$

- Finally, the *implicit* equation of a line in $\mathbb{R}^2$ is

$$ax + by + c = 0. \tag{13}$$

  All points whose coordinate vector $(x, y)$ satisfies this equation are on the line.

- The distance of a point $\mathbf{p}$ to a line $\mathbf{l}$ is the distance between $\mathbf{p}$ and the closest point on the line.
  - We can therefore derive the distance measurements.

- The normal form of the line is better for calculating distances, while the parametric form is better for generating points on the line.

## Plane and Hyperplanes

- In $\mathbb{R}^3$ and higher dimensions, the normal form for a line becomes a plane (hyperplane):

$$(\mathbf{x} - \mathbf{x}_0) \cdot \hat{\boldsymbol{\eta}} = 0. \tag{14}$$

  Here, we've assumed that the normal is a unit vector.

- In $\mathbb{R}^3$, a plane can also be specified by a point and two non-parallel direction vectors. This gives a parametric description. The cross-product of the direction vectors gives the normal.

- The parametric form of a point and direction is still a valid representation for a line in $\mathbb{R}^n$, and in fact the parametric definition above still holds.

## Important Inequalities

Here are two important inequalities that you are likely to already know:

- **Triangle:** If $\mathbf{a}$ and $\mathbf{b}$ are vectors in $\mathbb{R}^n$, then

$$\|\mathbf{a}+\mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|. \tag{15}$$

  This may be proven algebraically.

- **Schwarz:** If $\mathbf{a}$ and $\mathbf{b}$ are vectors in $\mathbb{R}^n$, then

$$|\mathbf{a} \cdot \mathbf{b}| \leq \|\mathbf{a}\| \cdot \|\mathbf{b}\|. \tag{16}$$

  This follows easily from (8).

## Lecture 1 Problems — Due Thursday, Jan 26th, in class

## Potential Test Questions

1. Prove that the dot product is commutative, associative and that $\mathbf{x} \cdot \mathbf{x}$ if and only if $\mathbf{x} = \mathbf{0}$. You will find it easiest to use the summation formula definition of the dot product.

2. Form two vectors, $\mathbf{u}$ and $\mathbf{v}$, of dimension $k$. Compute their dot-product, their projections onto each other, the angle between them, and convert them to unit vectors.

3. Give a proof of the triangle inequality.

4. Prove that two lines in $\mathbb{R}^2$ either intersect or are parallel to each other. Does this result hold in $\mathbb{R}^3$? Prove your answer.

## Questions for Gradings

1. (**5 points**) Find the intersection point of a line and a plane in $\mathbb{R}^3$ using any representation of lines and planes that you wish.

   **Solution:** Let the line be written parametrically as

   $$\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{d},$$

   and let the plane be the normal form,

   $$(\mathbf{x} - \mathbf{p}_0) \cdot \hat{\boldsymbol{\eta}} = 0,$$

   where $\mathbf{p}_0$ is a point on the plane. Substituting the first equation into the second and solving for $t$ yields

   $$t = \frac{\hat{\boldsymbol{\eta}}^\top (\mathbf{p}_0 - \mathbf{x}_0)}{\hat{\boldsymbol{\eta}}^\top \mathbf{d}}.$$

   Substituting yields the intersection point

   $$\mathbf{x} = \mathbf{x}_0 + \frac{\hat{\boldsymbol{\eta}}^\top (\mathbf{p}_0 - \mathbf{x}_0)}{\hat{\boldsymbol{\eta}}^\top \mathbf{d}} \mathbf{d}.$$

   In the special case that $\hat{\boldsymbol{\eta}}^\top \mathbf{d} = 0$, the line direction is perpendicular to the plane normal. Here, there is either no (finite) intersection or, when $\mathbf{x}_0 = \mathbf{p}_0$, the line is entirely contained in the plane.

2. (**10 points**) Prove that $\mathbf{x} \cdot (\mathbf{x} \times \mathbf{y}) = 0$ and give a geometric interpretation of the result.

   **Solution:** This is easy to prove by just writing the equation out in component form. The result is that $\mathbf{x}$ is perpendicular to its cross-section with any other vector.

3. (**15 points**) Given the parametric line equation

$$\mathbf{x}(t) = \mathbf{x}_0 + t\hat{\mathbf{d}} \tag{17}$$

where $\hat{\mathbf{d}}$ is the unit direction vector of the line. Find the closest point on the line to a given point $\mathbf{p}$. Give a geometric interpretation of the relationship between this closest point, the point $\mathbf{p}$, and the direction $\hat{\mathbf{d}}$. **Hint:** find the value of $t$ that minimizes the square distance

$$\|\mathbf{p} - \mathbf{x}(t)\|^2 \tag{18}$$

First solve the problem for two dimensions where $\mathbf{p} = (p_x, p_y)^T$ and $\mathbf{x}(t) = (x_0 + td_x, y_0 + td_y)$. Then try to solve it for $n$ dimensions. The latter requires that you use some vector algebra.

**Solution:** Here's the more general solution. You can get the special case in 2D by substituting in the coordinate form of the vectors. Expanding the above equation yields a square distance of

$$(\mathbf{p} - \mathbf{x}_0)^T(\mathbf{p} - \mathbf{x}_0) - 2t\hat{\mathbf{d}}^T(\mathbf{p} - \mathbf{x}_0) + t^2\hat{\mathbf{d}}^T\hat{\mathbf{d}}. \tag{19}$$

The $t^2\hat{\mathbf{d}}^T\hat{\mathbf{d}}$ in the third term reduces to $t^2$ because $\hat{\mathbf{d}}$ is a unit vector. Taking the derivative with respect to $t$ and setting the result equal to 0 produces

$$t = (\mathbf{p} - \mathbf{x}_0)^T\hat{\mathbf{d}}. \tag{20}$$

Note that the distance is minimized at this point rather than maximized because the distance equation is quadratic and goes to infinity as $t$ does. The closest point on the line is then

$$\mathbf{x}_c = \mathbf{x}_0 + [(\mathbf{p} - \mathbf{x}_0)^T\hat{\mathbf{d}}]\hat{\mathbf{d}}. \tag{21}$$

Geometrically, the vector from $\mathbf{p}$ to $\mathbf{x}_c$ is normal to the line direction $\hat{\mathbf{d}}$.