

**CSci 6974 and ECSE 6966 Math. Tech. for  
Vision, Graphics and Robotics**  
**Lecture 3, January 26, 2006**

**Vector Spaces and the Four Fundamental Spaces**

### Overview

Suppose we are sitting in the space  $\mathbb{R}^n$  and we are given a set of  $k$   $n$ -dimensional vectors. Call this set  $\mathcal{V} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ . We might ask and try to answer a series of questions:

1. What other vectors  $\mathbf{v}$  can we generate parametrically as a linear combination of the vectors in  $\mathcal{V}$ ? In other words what vectors  $\mathbf{v}$  may be written

$$\mathbf{v} = c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n$$

for some combination of scalar parameters  $\{c_1, \dots, c_n\}$ ?

2. Are all vectors  $\mathbf{v}_i \in \mathcal{V}$  needed, or can we throw some of them out and still generate the vectors  $\mathbf{v}$ ?
3. Clearly each generated vector  $\mathbf{v}$  is in  $\mathbb{R}^n$ , but can we generate every vector in  $\mathbb{R}^n$ ?
4. If we can't generate all of the vectors, what vectors can't we generate?
5. What linear combinations of vectors map back to the origin? In other words, what sets of scalars  $\{c_1, \dots, c_n\}$  are such that

$$c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n = \mathbf{0}$$

Clearly,  $\{0, \dots, 0\}$  does the trick, but are there other sets?

Understanding these questions leads to important definitions and properties of linear algebra.

### Vector Space — Definition

- Given a set of vectors  $W$ , a field of scalars  $F$ , and rules for vector addition and multiplication of vectors by scalars, we say that  $W$  is a vector space if for each pair of vectors  $\mathbf{u} \in W$  and  $\mathbf{v} \in W$  and for each scalar  $s \in F$ ,

$$\mathbf{u} + \mathbf{v} \in W$$

and

$$s\mathbf{u} \in W$$

- Generally, we will take  $F$  to be the field of real numbers,  $\mathbb{R}$ , and  $W$  to be a set of vectors in  $\mathbb{R}^n$ .

- Aside: A field has addition and multiplication operations that satisfy the usual commutativity, associativity, distributivity, identity, and inverse axioms.
- Formally, the rules for vector addition and scalar multiplication must be defined appropriately, but for our purposes, the usual definitions for  $\mathbb{R}^n$  will suffice.

## Subspaces

- **Definition:** A subspace of a vector space is a set  $S \subseteq W$  that is closed under vector addition and scalar multiplication.
- What vector must be in any subspace?

## Linear Combinations

- Given vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k$  and scalars  $c_1, \dots, c_k$ , the vector

$$\mathbf{v} = c_1\mathbf{v}_1 + \dots + c_k\mathbf{v}_k$$

is obtained as a *linear combination* of the vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k$ .

- Can you write this using matrix notation?
- The set of all vectors  $\mathbf{v}$  obtainable in this way is said to be the *span* of  $\mathbf{v}_1, \dots, \mathbf{v}_k$ .

## Linear Independence, Basis and Dimension

- A set of vectors  $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  is *linearly independent* if

$$c_1\mathbf{v}_1 + \dots + c_k\mathbf{v}_k = \mathbf{0} \quad \text{implies } c_1 = c_2 = \dots = c_k = 0.$$

In other words, all non-trivial linear combinations of the vectors are non-zero.

- Otherwise,  $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  is *linearly dependent*, which means that at least one vector  $\mathbf{v}_i$  in this set may be written as a linear combination of the remaining vectors.
  - Removing  $\mathbf{v}_i$  from the set does not change its span.
- A set of linearly independent vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k$  is a *basis* for vector space  $W$  if each  $v \in W$  may be written as a linear combination of  $\mathbf{v}_1, \dots, \mathbf{v}_k$ , i.e if the set spans  $W$ .
- All bases for  $W$  have the same number of vectors.
- The dimension of vector space  $W$  is the size of any one of its sets of basis vectors.

## Row Spaces, Column Spaces and Nullspaces

All of the above questions can be translated into questions about matrices by interpreting the columns  $\mathbf{a}_1, \dots, \mathbf{a}_n$  of a matrix  $\mathbf{A}$  as the vectors  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  described above. Doing this leads to questions about some fundamental properties of a matrix.

- Four fundamental spaces are associated with a given matrix  $\mathbf{A}$ .

**Row space** is the span of the rows of  $\mathbf{A}$ .

**Column space** is the span of the columns of  $\mathbf{A}$ , as above.

**Nullspace** is the set of vectors  $\mathbf{x}$  such that

$$\mathbf{Ax} = \mathbf{0}.$$

**Left nullspace** is the set of vectors  $\mathbf{y}$  such that

$$\mathbf{y}^T \mathbf{A} = \mathbf{0}.$$

- The dimension of the row space and the dimension of the column space are each equal to the rank of  $\mathbf{A}$ . Denote this by  $r$ .
- The dimension of the (right) nullspace is  $n - r$ .
- The dimension of the left nullspace is  $m - r$ .
- The row-space is orthogonal to the nullspace and the column-space is orthogonal to the right nullspace. These properties are easy to prove.

## Practice Problems / Potential Test Questions

1. Prove that the zero vector,  $\mathbf{0}$ , must be in any subspace of  $\mathbb{R}^n$ .
2. Find sets of basis vectors for the row space, the column space and the left and right nullspaces of

$$\begin{pmatrix} 0 & 1 & -2 \\ 1 & 4 & -7 \\ 3 & 2 & -1 \\ -2 & 5 & -12 \end{pmatrix}$$

3. What is the smallest subspace of  $3 \times 3$  matrices (of real numbers) that contains all symmetric matrices and all lower triangular matrices? What is the largest subspace that contains all symmetric matrices and all lower triangular matrices? Justify your answers.

## HW 2: Problems For Grading

Submit solutions to the following problems on Monday, Jan 30th.

1. **(15 points)** Which of the following subsets of  $\mathbb{R}^3$  are actually subspaces? Justify your answers.

- (a) The plane of vectors  $\mathbf{v}^\top = (x, y, z)$  with  $x = 0$ .
- (b) The vectors  $\mathbf{v} = (x, y, z)$  such that  $xy = 0$ .
- (c) The vectors  $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$  that satisfy

$$\mathbf{v}_3 - \mathbf{v}_2 + 3\mathbf{v}_1 = 0.$$

2. **(15 points)** If I exchange two rows of a matrix  $\mathbf{A}$ , which of the four fundamental spaces remains fixed and which change (in general). Prove your answer.