CSci 6974 and ECSE 6966 Math. Tech. for Vision, Graphics and Robotics
Lecture 4, January 30, 2006
LU Decompositions

Systems of Linear Equations

We are going to take a slight detour to solve a 3×3 system of linear equations, and we will do so without using matrices. Here is the first example:

$$
\begin{aligned}
x + y + z &= 4 \\
-2x + y + 4z &= -5 \\
-x + 3y + 4z &= 3
\end{aligned} \tag{1}
$$

Row Manipulation As Matrix Multiplication

- We will rewrite the operations as a matrix manipulation.
- When solving $\mathbf{Ax} = \mathbf{b}$, this will lead to
$$\mathbf{L'Ax} = \mathbf{L'b} \tag{2}$$
where $\mathbf{L'A} = \mathbf{U}$. Here $\mathbf{L'}$ is lower triangular and $\mathbf{U}$ is upper triangular.
- $\mathbf{L'}$ will be invertible by construction. We will write $\mathbf{L} = \mathbf{L'}^{-1}$.
- This leads to the "LU" decomposition:
$$\mathbf{A} = \mathbf{LU} \tag{3}$$
- The only caveat is that $\mathbf{A}$ may require a row permutation.

Finding the 4 Fundamental Spaces

- We'll now see how the above "row" operations can be used to find the row and column spaces and null spaces of a matrix.
- We'll use the example of the non-square matrix:
$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{pmatrix}. \tag{4}$$
- This will lead to a decomposition of the form:
$$\mathbf{L'A} = \mathbf{U} \qquad \text{where} \qquad \mathbf{U} = \begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & 0 & u_{23} & u_{24} \\ 0 & 0 & 0 & 00 \end{pmatrix} \tag{5}$$
where the fundamental result is that $u_{11} \neq 0$ and $u_{23} \neq 0$.

- From this example we will argue that:
  - The first two rows of $\mathbf{U}$ form a basis for the row space.
  - The first and third columns of $\mathbf{A}$ form a basis for the column space.
  - The (right) nullspace of $\mathbf{A}$ consists of vectors of the form $(0, x_2, 0, x_4)^\top$.

Proving Other Properties

We may use the row operations to prove a variety of other properties:

- The dimensions of the row and column spaces of a matrix are equal.
- The rank of a matrix is the dimension of the row space or the column space. If $\mathbf{A}$ is $m \times n$, then $\text{rank}(\mathbf{A}) \leq \min(m, n)$.
- Any two basis sets for a vector space must have the same number of vectors.
- A $m \times n$ matrix $\mathbf{A}$ with $m < n$ has a right inverse if and only if its rank is m. This right inverse is

$$\mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}. \tag{6}$$

- A $m \times n$ matrix $\mathbf{A}$ with $m > n$ has a left inverse if and only if its rank is n. This left inverse is

$$(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T. \tag{7}$$

Practice Problems / Potential Test Questions

1. Given a square matrix $\mathbf{A}$, a permutation matrix $\mathbf{P}$ is a square matrix such that $\mathbf{PA}$ interchanges rows of $\mathbf{A}$.

   (a) For a 4×4 matrix $\mathbf{A}$, write a permutation matrix that moves the 1st row to the 3rd row, the 3rd row to the 4th row, and the 4th row to the 1st, leaving the second row unchanged.

   (b) Show that this does not change the row-space of $\mathbf{A}$.

2. Find sets of basis vectors for the row space, the column space and the left and right nullspaces of
$$\begin{pmatrix} 0 & 1 & -2 \\ 1 & 4 & -7 \\ 3 & 2 & -1 \\ -2 & 5 & -12 \end{pmatrix}$$

Problems For Grading

Submit solutions to the following problems on Monday, February 6th, as part of HW 3.

1. (**15 points**) Suppose have a set of k points, $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ in $\mathbb{R}^n$. Using the methods described in these notes, outline a procedure for determining if these points are coplanar. Hint: in order to form a subspace, a plane must go through the origin (as proven above), but the origin can be shifted to go through any point in $\mathbb{R}^n$.