CSci 6974 and ECSE 6966 Math. Tech. for Vision, Graphics and Robotics
Lecture 5, February 3, 2006
Basis, Orthogonality, Factorization

Summary of Procedure for Finding Spaces

Given $m \times n$ matrix $\mathbf{A}$:

1. Find permutation matrix $\mathbf{P}$, lower triangular matrix $\mathbf{L}$, and "echelon matrix $\mathbf{U}$, such that
$$\mathbf{PA} = \mathbf{LU}. \tag{1}$$
2. The non-zero rows of $\mathbf{U}$ form a basis for the row-space of $\mathbf{A}$.
3. The columns of $\mathbf{A}$ corresponding to the pivots of $\mathbf{U}$ (the first non-zero entry in each row) form a basis for the column space of $\mathbf{A}$.
4. Examine the equation $\mathbf{Ux} = \mathbf{0}$. Write $\mathbf{x} = (x_1, \ldots, x_n)^\top$. The entries x_j corresponding to the columns of $\mathbf{U}$ without pivots are "free variables"; the other variables are non-free, or echelon variables. Repeat the following for each free variable x_j:
   (a) Form $\mathbf{x}$ with $x_j = 1$, the entries for the other free variables $= 0$, and the nonpivot (echelon) variables as unassigned variables.
   (b) Solve $\mathbf{Ux} = \mathbf{0}$ for $\mathbf{x}$ using back-substitution for the values of the echelon variables.
   (c) The resulting $\mathbf{x}$ is a basis vector for the null-space.

   Combining all resulting vectors gives a basis for the null-space.

Motivation

Thus far we have produced a set of basis vectors for the row, column and null spaces through the LU decomposition, but have done nothing about the properties of these vectors. For example, the pair of vectors $\mathbf{v}_1^\top = (1, 0, 1)$ and $\mathbf{v}_2^\top = (0.9, 0.1, 1.1)$ forms a basis for a plane through the origin in $\mathbb{R}^3$, but the two vectors are close to being parallel.

Orthogonality and Orthonormal Bases

- A set of basis vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_k$, is *orthonormal* if
$$\mathbf{v}_i^T \mathbf{v}_j = \delta_{ij}$$
  for all i, j. Here, δ_{ij} is the Kronecker delta function introduced in the Lecture 2 notes.

- An *orthonormal* basis is a basis whose vectors are orthonormal.
- What is a simple orthonormal basis for $\mathbb{R}^n$? Can you come up with some others?
- How can you write a vector $\mathbf{a}^T = (a_1, \ldots, a_n)$ in terms of an orthonormal basis, $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$?

Orthogonal Matrices

- An $n \times n$ square matrix, $\mathbf{Q}$, is *orthogonal* if its column vectors are orthonormal.
- Immediately, we have
$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I}.$$
- The uniqueness of the inverse for a square matrix implies
$$\mathbf{Q}\mathbf{Q}^T = \mathbf{I}.$$
- Putting these together means that $\mathbf{Q}^{-1} = \mathbf{Q}^T$.
- The determinant of an orthogonal matrix is either 1 or -1.
- Rotation matrices are orthogonal matrices with determinant 1.

Gram-Schmidt Orthogonalization

- Any set of basis vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ can be converted to an orthonormal basis using the following Gram-Schmidt orthogonalization process.
- The process is iterative:
  1. $\mathbf{v}_1 = \mathbf{a}_1$.
  2. $\mathbf{q}_1 = \mathbf{v}_1 / \|\mathbf{v}_1\|$.
  3. For $i \geq 2$:
  $$\mathbf{v}_i = \mathbf{a}_i - \sum_{j=1}^{i-1} (\mathbf{a}_i^T \mathbf{q}_j)\mathbf{q}_j$$
  and
  $$\mathbf{q}_i = \frac{\mathbf{v}_i}{\|\mathbf{v}_i\|}$$
- Note that this is done in terms of the orthogonal projections of vectors that we discussed in Lecture 1.
- It is straight-forward to show that the $\mathbf{q}_1, \ldots, \mathbf{q}_n$ are orthonormal.

QR Factorization

The Gram-Schmidt process just described may be converted to a matrix factorization method.

- Any square matrix **A** may be factored in any one of the following ways:
  - $\mathbf{A} = \mathbf{QR}$, where **Q** is orthogonal and **R** is upper triangular ("right"),
  - $\mathbf{A} = \mathbf{RQ}$, where **R** is upper triangular and **Q** is orthogonal,
  - $\mathbf{A} = \mathbf{QL}$, where **Q** is orthogonal and **R** is lower triangular ("left"),
  - $\mathbf{A} = \mathbf{RQ}$, where **L** is lower triangular and **Q** is orthogonal,
- The notation is unfortunate, because later we will want to use the matrix name **R** for orthogonal (rotation) matrices.

Matrix Factorization Methods

- There are many different matrix factorization / decomposition methods:
  - LDU decomposition
  - QR factorization
  - Cholesky factorization (some matrices)
  - Spectral (eigenvalue) decomposition (some matrices)
  - Singular value decomposition
- Some of these only apply to restricted types of matrices.
- The different decompositions make explicit types of matrix properties and are therefore useful in a variety of circumstances.

Practice Problems / Potential Test Questions

1. Outline computational solutions to the problems posed at the start of Lecture 3.

2. Using Gram-Schmidt form an orthonormal basis from

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}.$$

   Form the QR decomposition of $\mathbf{A} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix}$.

3. What happens during Gram-Schmidt when the vectors are linearly dependent? Justify your answer.

4. Suppose you are given a set of n-dimensional vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_k$. Outline a procedure for converting this set to an orthonormal basis for $\mathbb{R}^n$. You may assume that $\mathbf{v}_1 \neq \mathbf{0}$, but you ca not assume the vectors are linearly independent and you can't assume their span is $\mathbb{R}^n$.

5. Prove that the determinant of an orthogonal matrix is either 1 or -1.

Problems For Grading

Submit solutions to the following problems on Monday, February 6th, as the second part of HW 3.

1. (**15 points**) Suppose have a set of k points, $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ in $\mathbb{R}^n$. Using the methods described in these notes, outline a procedure for determining if these points are coplanar. Hint: in order to form a subspace, a plane must go through the origin (as proven above), but the origin can be shifted to go through any point in $\mathbb{R}^n$.

2. (**10 pts**) Give formulii for the $|\det(\mathbf{A})|$ for square matrix $\mathbf{A}$ from its

   (a) **QR** factorization,

   (b) Cholesky decomposition (when $\mathbf{A}$ is positive definite).

   Justify your answers.