CSci 6974 and ECSE 6966 Math. Tech. for Vision, Graphics and Robotics
Lecture 6, February 6, 2006
Eigenvalues and Eigenvectors

Overview

Eigenvalues and eigenvectors appear in many problems. Here are just two that we will be interested in:

- What are the "fixed points" of a transformation — i.e. what points, lines or directions remain unchanged by the transformation?
- How stable is a solution to a minimization problem?

Eigenvalues and Eigenvectors

Eigenvalues and eigenvectors are only defined for square matrices.

- The vector $\mathbf{x}$ is an *eigenvector* of matrix $\mathbf{A}$ and the scalar λ is an *eigenvalue* of $\mathbf{A}$ if
$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}.$$
- We can show that equivalently, the scalar λ is an eigenvalue of $\mathbf{A}$ if
$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0.$$
(This is the *characteristic equation* for $\mathbf{A}$.) and $\mathbf{x}$ is an eigenvector of $\mathbf{x}$ if $\mathbf{x}$ is in the nullspace of $\mathbf{A} - \lambda\mathbf{I}$.

Example: Rotation

Consider the following rotation matrix:

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- This matrix has 1 real and 2 complex eigenvalues.
- The eigenvector corresponding to the real eigenvalue is $(0, 0, 1)^T$, which is the z-axis
- Points along this axis are unchanged by the rotation, and therefore these are "fixed points" of the rotation.
- All rotations in $\mathbb{R}^3$ have this same structure: a single fixed axis — the axis of rotation.

Properties of Eigenvalues and Eigenvectors

- Suppose $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $\mathbf{A}$, then

$$\sum \lambda_i = \text{trace}(\mathbf{A}), \qquad \text{and}$$
$$\prod \lambda_i = \det(\mathbf{A}).$$

- For a triangular matrix, the eigenvalues are the diagonal elements.

- $\mathbf{A}$ and $\mathbf{A}^2$ have the same eigenvectors and if λ_i is an eigenvalue of $\mathbf{A}$, then λ_i^2 is an eigenvalue of $\mathbf{A}^2$.

- If $\mathbf{A} = \mathbf{M}^{-1}\mathbf{B}\mathbf{M}$ for invertible matrix $\mathbf{M}$, then $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvalues, but not the same eigenvectors.

Computing Eigenvalues and Eigenvectors

- There is no simple form for computing eigenvalues and eigenvectors because there is no expression for the roots of a quintic or higher polynomial...

- Suppose $n \times n$ matrix $\mathbf{A}$ has a linearly independent set of eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$, with associated eigenvalues $\lambda_1, \ldots, \lambda_n$. Form the matrix

$$\mathbf{P} = \big(\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n\big).$$

  Then,

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \text{diag}(\lambda_1, \ldots \lambda_n).$$

  Eigenvalue / eigenvector problems are generally solved by constructing such a matrix $\mathbf{P}$ for a given matrix $\mathbf{A}$.

Symmetric Matrices and Spectral Decompositions

- The eigenvalues of symmetric matrices are all real, though not necessarily positive. This is easy to prove and we will do so in class.

- The eigenvectors corresponding to distinct eigenvalues of a symmetric matrix are orthogonal. You will have to prove this as a homework problem!

- Any symmetric $n \times n$ matrix $\mathbf{A}$ can be written as

$$\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^T = \sum_{i=1}^{n} \lambda_i \mathbf{v}_i \mathbf{v}_i^T,$$

  where $\mathbf{V}$ is an orthogonal matrix whose columns are normalized eigenvectors $\mathbf{v}_i$ of $\mathbf{A}$, and $\mathbf{D}$ is a diagonal matrix containing the corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$.

Quadratic Forms and Positive Definite Matrices

- For symmetric matrix $\mathbf{A}$, the expression

  $$\mathbf{x}^T \mathbf{A} \mathbf{x}.$$

  is called a *quadratic form*.

- Matrix $\mathbf{A}$ is said to be *positive definite* if

  $$\mathbf{x}^T \mathbf{A} \mathbf{x} > 0 \qquad \text{for all non-zero } \mathbf{x},$$

  or, equivalently, all eigenvalues of $\mathbf{A}$ are positive.

- If all eigenvalues are non-negative or, equivalently, $\mathbf{x}^T \mathbf{A} \mathbf{x} \geq 0$, then $\mathbf{A}$ is *positive semi-definite*.

- Any matrix of the form $\mathbf{A} = \mathbf{B}^T \mathbf{B}$ is positive semi-definite.

- When the *Hessian* matrix of a function minimization problem is positive definite, the solution is stable.

Practice Problems / Potential Test Questions

1. Suppose that $\mathbf{A}$ and $\mathbf{B}$ are square matrices and $\mathbf{v}$ is an eigenvector of $\mathbf{A}$. We know that $\mathbf{v}$ is also an eigenvector of $\mathbf{A}^2$. Is $\mathbf{v}$ also an eigenvector of $\mathbf{C} = \mathbf{BA}$? Prove your answer.

2. Prove that the eigenvectors of distinct eigenvalues of a 2×2 matrix are linearly independent.

3. Show that

$$\mathbf{x}^T \mathbf{A} \mathbf{x} > 0$$

   for symmetric matrix $\mathbf{A}$ and all vectors $\mathbf{x}$ implies that all eigenvalues of $\mathbf{A}$ are positive.

Problems For Grading

Submit solutions to the following problems on Monday, February 13th, as the first part of HW 4.

1. (**10 points**) Prove that the eigenvectors of distinct eigenvalues of a symmetric matrix are orthogonal to each other.

2. (**10 points**) How can you find the nullspace of a square matrix from its eigenvalues and eigenvectors?