CSci 6974 and ECSE 6966 Math. Tech. for Vision, Graphics and Robotics
Lecture 9, February 16, 2006
Principal Component Analysis and Its Applications

Overview of Today's Lecture

- Finish discussion of orthogonal least-squares, including an important alternative derivation to the one presented in the Lecture 8 notes.
- PCA and its application to recognition and shape modeling.

Orthogonal Regression — Alternative Form

- A second formulation for orthogonal regression writes the error term as a function of the plane normal $\boldsymbol{\eta}$ vector and a point $\mathbf{x}_0$ on the plane:

$$E(\boldsymbol{\eta}, \mathbf{x}_0) = \sum_{i=1}^{N} [\boldsymbol{\eta}^\top (\mathbf{x}_i - \mathbf{x}_0)]^2, \tag{1}$$

  Of course, in this formulation, $\mathbf{x}_0$ is not unique.

- The first step is to show that one choice for $\mathbf{x}_0$ is the center of mass of the points. This is a homework problem.
- The problem then reduces to finding the unit vector $\boldsymbol{\eta}$ minimizing

$$E(\boldsymbol{\eta}) = \boldsymbol{\eta}^\top \left(\sum_{i=1}^{N} \mathbf{x}'_i {\mathbf{x}'_i}^\top \right) \boldsymbol{\eta} \tag{2}$$

  with $\mathbf{x}'_i = \mathbf{x}_i - \mathbf{x}_0$.

- Once again we can determine $\boldsymbol{\eta}$ using either the SVD or the spectral decomposition.

Eigenimages and Recognition

- Consider a collection of images of one or more faces. The problem is to take a new image and quickly find the image closest to it in the collection.
- We will assume the faces are centered in the images, the images are all the same size, and the intensities are centered and normalized.
- Each image is then converted to a vector by reading out the pixel values in row-major order. For image i, let $\mathbf{x}_i$ be the associated vector.

- Store the vectors in the columns of a matrix:

$$\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_n). \tag{3}$$

If there are m pixels in the images, this matrix is $m \times n$, with $m \gg n$.

- Compute either the singular value decomposition of $\mathbf{X}$ or the spectral decomposition of $\mathbf{X}\mathbf{X}^\top$, i.e.

$$\mathbf{X} = \mathbf{U}\mathbf{D}\mathbf{V}^\top \qquad \text{or} \qquad \mathbf{X}\mathbf{X}^\top = \mathbf{U}\mathbf{D}^2\mathbf{U}^\top. \tag{4}$$

- The first few singular values (eigenvalues) are the largest. The associated columns of $\mathbf{U}$ give the directions of image space (a subspace of $\mathbb{R}^m$) along which the images vary most substantially.

- The images are stored by storing the first k columns of $\mathbf{U}$ — call the matrix $\mathbf{U}_k$ — and then storing the projection of each $\mathbf{x}_i$ onto $\mathbf{U}_k$, i.e.,

$$\mathbf{c}_i = \mathbf{U}_k^\top \mathbf{x}_i. \tag{5}$$

- Each image is therefore represented as a k-component vector, a greatly-reduced representation.

- Another image vector (following centering and normalization), $\mathbf{x}$, is compared to the stored images by computing

$$\mathbf{c} = \mathbf{U}_k^\top \mathbf{x}. \tag{6}$$

and comparing the $\mathbf{c}$ against the stored $\mathbf{c}_i$.

- Artificial face images may be created by generating a vector $\mathbf{c}$ and then generating an image

$$\mathbf{x} = \mathbf{U}_k \mathbf{c}. \tag{7}$$

By keeping the components of $\mathbf{c}$ within (about) ± 2.5 times the corresponding singular values, reasonable synthetic faces may be generated.

- Some of these ideas are illustrated in the example is included with these notes.

Active Shape Models

- We can achieve similar results with the position of points from a deformed 2d or 3d shape. One example is the position of points on a hand.

- Suppose the shape is represented by the position of the m 2d points (x_j, y_j). Let

$$\mathbf{a}_i^\top = (x_{i,1}, y_{i,1}, x_{i,2}, y_{i,2}, \ldots, x_{i,m}, y_{i,m}) \tag{8}$$

stored the position of all m points in the i^{th} instance of the shape.

- We can combine these positions into a matrix

$$\mathbf{A} = (\mathbf{a}_1, \ldots, \mathbf{a}_n). \tag{9}$$

  Here there are n time instances, and $\mathbf{A}$ is $2m \times n$.

- Following centering and normalization, we can compute the singular value decomposition

$$\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^\top. \tag{10}$$

- Once again the first few vectors of $\mathbf{U}$ give the major directions along which the $\mathbf{a}_i$ values. Call the matrix $\mathbf{U}_k$.

- We can store a reduced version of the $\mathbf{a}_i$'s by projecting:

$$\mathbf{b}_i = \mathbf{U}_k^\top \mathbf{a}_i. \tag{11}$$

- We can recognize if a configuration of points $\mathbf{a}$ is consistent with the shape by computing

$$\mathbf{b} = \mathbf{U}_k^\top \mathbf{a}. \tag{12}$$

  If the values of $\mathbf{b}$ are within a few multiples of the singular values then $\mathbf{a}$ is reasonably close to the shape.

- Once again we can generate representative shapes as well.

Concluding Comments

- The analysis we have done is an application of what is called "Principal Component Analysis" (PCA) in the statistics and estimation literature.

- In general, we do not compute the entire SVD of $\mathbf{X}$ or $\mathbf{A}$. There are efficient algorithms, such as the Karhunen-Loéve transform for finding the first k singular values and associated vectors of $\mathbf{U}$.

- In the recognition and shape generation applications, the above techniques have provided the basis for more advanced and ongoing research in vision and graphics.

Homework 5 Problem

Solutions to the following problem are due Tuesday, February 21 in class.

1. **(10 points)** Given the alternative formulation for orthogonal least-squares,

$$E(\boldsymbol{\eta}, \mathbf{x}_0) = \sum_{i=1}^{N} [\boldsymbol{\eta}(\mathbf{x}_i - \mathbf{x}_0)]^2,$$

show that one value of the point $\mathbf{x}_0$ is the center of mass (the average) of the $\mathbf{x}_i$ values.