CSci 6974 and ECSE 6966 Math. Tech. for Vision, Graphics and Robotics
Lecture 10, February 21, 2006
Introduction to Projective Geometry

Overview of Today's Lecture

- Shortened discussion from Lecture 9 of PCA and its application to recognition and shape modeling.
- Introduction to projective geometry in 2D.
  - This material is covered in Chapter 1 of Hartley and Zisserman, although we are motivating it differently here.
  - The emphasis is homogeneous representations.
  - Lecture 11 will cover projective transformations.

Vanishing Points

- A simplified formulation of the perspective (pinhole) projection of a point $(x, y, z)^\top$ is

$$(u, v)^\top = \begin{pmatrix} \frac{fx}{z} & \frac{fy}{z} \end{pmatrix}^\top, \tag{1}$$

  where f is the focal length of the camera. This can be demonstrated using similar triangles assuming that all projection line pass through the origin and intersect the camera plane at $z = -f$.

- Recall that the parametric equation of a line in the world is

$$\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{d} \tag{2}$$

  where $\mathbf{x}_0 = (x_0, y_0, z_0)^\top$ and $\mathbf{d} = (d_x, d_y, d_z)^\top$.

- The parametric image equation for the projection of this line is

$$\big(u(t), v(t)\big) = \left(\frac{f(x_0 + td_x)}{z_0 + td_z}, \frac{f(y_0 + td_y)}{z_0 + td_z} \right). \tag{3}$$

  As you are asked to prove in the homework, as t varies this really does form a line in 2D!

- Now consider what happens to the image point as $t \to \infty$.
  - We get the image point

$$\left(\frac{fd_x}{d_z}, \frac{fd_y}{d_z} \right) \tag{4}$$

  - The image projection of the limit is a real image point, called a "vanishing point," despite the fact that the limiting point in the world is infinitely far from the camera.

- Notice that any other line parallel to this one has the exact same vanishing point.
- We need a geometry that is able to describe "points at infinity" — intersections of parallel lines — and horizon lines, which are "lines at infinity".
- Projective geometry allows us to do this.
- We'll start with two-dimensional projective geometry — the space $\mathcal{P}^2$.

Lines and Points; Homogeneous Representations

- As we have already discussed this semester, the line equation,

$$ax + by + c = 0, \tag{5}$$

  can be written as the dot product:

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0 \tag{6}$$

- The vector of line parameters $\mathbf{l} = (a, b, c)^\top$ is unique only up to a (non-zero) scale factor.
- We can also write the point as a vector $\mathbf{x} = (x, y, 1)^\top$, and then write $\mathbf{x} = (\lambda x, \lambda y, \lambda)^\top$, for $\lambda \neq 0$. This is the same point.
- More generally we write $\mathbf{x} = (x_1, x_2, x_3)^\top$.
  - This is the *homogeneous* representation of the point.
  - We can recover the original x and y from the ratios x_1/x_3, x_2/x_3. The values x and y are the associated *affine coordinates* of the point.
- The line equations are now simply

$$\mathbf{l}^\top \mathbf{x} = \mathbf{x}^\top \mathbf{l} = 0. \tag{7}$$

Intersections and Duality

- For two distinct vectors of line parameters $\mathbf{l}_1$, $\mathbf{l}_2$, we can find the unique point on both lines from

$$\mathbf{x} = \mathbf{l}_1 \times \mathbf{l}_2, \tag{8}$$

  or equivalently from the null space of

$$\mathbf{L} = \begin{pmatrix} \mathbf{l}_1^\top \\ \mathbf{l}_2^\top \end{pmatrix}. \tag{9}$$

- We can do the exact same thing to find the vector describing the line joining two points.
- These properties arise from the homogeneous equation $\mathbf{l}^\top \mathbf{x} = 0$ and from the fact that we have written $\mathbf{x}$ in homogeneous form.
- *Duality*: The symmetry between points and lines in all of the above expressions is exact. Any theorem in $\mathcal{P}^2$ about lines is true about points and vice-versa.

Points at Infinity; Line at Infinity

- The intersection point between $\mathbf{l}_1 = (a, b, c)^\top$ and $\mathbf{l}_2 = (a, b, d)^\top$ is $\mathbf{x} = (-b, a, 0)^\top$.
- Points with 0 as their last component are "points at infinity". These points have no affine analog.
- There is a 1-1 correspondence between line directions and points at infinity in $\mathcal{P}^2$.
- The line $\mathbf{l}_\infty = (0, 0, 1)^\top$ is the "line at infinity". All points at infinity are on this line.
- In projective geometry there is nothing special about points and lines at infinity. Only when we try to map them into an affine world do they become special (non-mappable).

Conics

- You probably learned the following general equation of a conic
$$ax^2 + bxy + cx^2 + dx + ey + f = 0. \tag{10}$$
- This can be written as a quadratic form involving a symmetric matrix:
$$\begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0, \tag{11}$$
or more simply,
$$\mathbf{x}^\top \mathbf{C} \mathbf{x}, \tag{12}$$
where $\mathbf{C}$ is symmetric and rank 3.
- Five points determine a conic. We can find this conic (the parameter vector $(a, b, c, d, e, f)^\top$ from the null space of
$$\begin{pmatrix} x_1^2 & x_1 y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2 y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3 y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4 y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5 y_5 & y_5^2 & x_5 & y_5 & 1 \end{pmatrix}. \tag{13}$$

- Although it is liikely that the conics you have dealt with in the past create rank 3 conic matrices, conic matrices of rank less than 3 are sometimes of interest.
- The line $\mathbf{l} = \mathbf{C}\mathbf{x}$ is tangent to the conic at point $\mathbf{x}$.
- There is the notion of a "dual" or line conic for a given (point) conic. This conic is $\mathbf{C}^*$ — the adjoint of $\mathbf{C}$. If $\mathbf{C}$ is rank 3, $\mathbf{C}^*$ equals $\mathbf{C}^{-1}$, up to an irrelevant scale factor.
- Line $\mathbf{l}$ is tangent to $\mathbf{C}$ if and only if

$$\mathbf{l}^\top \mathbf{C}^* \mathbf{l} = 0. \tag{14}$$

Homogeneous Forms and Scale Factors

- One of the hardest things to get used to in projective geometry is that representations are no longer unique. In particular:
  - Two points, $\mathbf{x}_1$ and $\mathbf{x}_2$, are equal if $\mathbf{x}_1 = \lambda \mathbf{x}_2$ for real number $\lambda \neq 0$.
  - Two lines, $\mathbf{l}_1$ and $\mathbf{l}_2$, are equal if $\mathbf{l}_1 = \lambda \mathbf{l}_2$ for real number $\lambda \neq 0$.
  - Two conics, $\mathbf{C}_1$ and $\mathbf{C}_2$, are equal if $\mathbf{C}_1 = \lambda \mathbf{C}_2$ for real number $\lambda \neq 0$.
- These arise from our desire to represent points and lines at infinity and because all equations have a homogeneous form:

$$\mathbf{l}^\top \mathbf{x} = 0 \qquad \text{and} \qquad \mathbf{x}^\top \mathbf{C} \mathbf{x} = 0.$$

  - One implication of this is that a conic matrix only has 5 degrees of freedom.
- Counting degrees of freedom in projective geometry and in other problems can be tricky. It is always useful however as (a) a sanity check, and (b) to verify your understanding.

Practice Problems and Potential Test Questions

1. Find the intersection point of lines $\mathbf{l}_1 = (1,0,5)^\top$ and $\mathbf{l}_2 = (-3,2,4)$. What are the affine coordinates of this point?

2. Show that the null space of a 2×3 matrix is equal to the cross-product of the two row vectors.

3. At first, parametric equations of lines in $\mathcal{P}$ "feel funny". But, parametric equations can be written in the usual way as a linear combinations of two points. In particular, if $\mathbf{x}$ and $\mathbf{y}$ are points on the line and $\mathbf{d}$ is the (two component) tangent direction, then the line can be written as any one of the following:

$$\mathbf{p}(\alpha) = \mathbf{x} + \alpha\mathbf{y}$$
$$\mathbf{p}(\beta) = \beta\mathbf{x} + (1-\beta)\mathbf{y}$$
$$\mathbf{p}(\gamma) = \mathbf{x} + \gamma \begin{pmatrix} \mathbf{d} \\ 0 \end{pmatrix}$$

Prove that these forms are all equivalent. To do this, recall the conditions under which two points $\mathbf{x}_1$ and $\mathbf{x}_2$ are equal and recall the relationship between points at infinity and line directions.