CSci 6974 and ECSE 6966 Math. Tech. for Vision, Graphics and Robotics
Lecture 11, February 27, 2006
Projective, Affine and Similarity Transformations

Overview of Today's Lecture

- Complete the discussion from Lecture 10 on homogeneous representations, including the line at infinity and conics.
- Transformations: projective, similarity, affine.
- Undoing distortions

Projective Transformations

The study of transformations and their properties (especially their invariants) is fundamental to geometry.

- A projective transformation is an invertible mapping that preserves colinearity
- Projective transformations are also known as homographies, projectivities and colineations.
- One may prove (not easily) that projective transformations may be represented by a 3×3 invertible matrix:

  $$\mathbf{x}' = \mathbf{H}\mathbf{x} \tag{1}$$

  Note that $\mathbf{H}$ is determined up to an arbitrary scale factor so it has 8 degrees of freedom.
- We are interested in these transformations for 2 primary reasons, one theoretical and one practical:
  - Homographies are the most general linear transformation, and rigid and affine transformations are special cases of homographies.
  - As we will prove in a later lectures, the perspective projection of a planar surface in the world reduces to a plane homography. The composition of two homographies is a homography. Thus, two different images of a planar surface are related by a homography.
  - Even though the matrix $\mathbf{H}$ is 3×3, this is still a transformation of two-dimensional objects because everything is done using homogeneous coordinates.

Example: Computing Projective Transformations

Suppose we have an image of the face of a building that has severe perspective distortions. If we know, for example, the dimensions of a window or door that appears in the image, we can compute a transformation that removes these distortions.

- Suppose the width of the door is w and the height is h. Let the ratio be $r = h/w$.
- Find the four corners in the image. Call the locations $\mathbf{x}_1$ through $\mathbf{x}_4$.
- Designate four locations for these to map to: $(0, 0, 1)^T$, $(1, 0, 1)^T$, $(0, r, 1)^T$ and $(1, r, 1)^T$. Note that we are creating an arbitrary Euclidean coordinate in the image.
- These four "correspondences" allow us to compute a unique transformation, $\mathbf{H}$.
- We apply this transformation to the entire image. It undoes the effects of the projective distortion.

Projective Transformations of Lines and Conics

- Using the basic result $\mathbf{x}' = \mathbf{H}\mathbf{x}$ we can calculate the perspective transformations of lines and conics.
- Lines transform as $\mathbf{l}' = \mathbf{H}^{-T}\mathbf{l}$, where $\mathbf{H}^{-T}$ is the inverse of the transpose of $\mathbf{H}$.
- Conics transformation as $\mathbf{C}' = \mathbf{H}^{-T}\mathbf{C}\mathbf{H}^{-1}$.
- Dual conics transformation as $\mathbf{C}^{*\prime} = \mathbf{H}\mathbf{C}^{*}\mathbf{H}$.
  - You are asked to proved this last property as a practice problem.

Transformation Groups

- The set of projective transformations, together with matrix multiplication, forms a group.
  - A group is a set of elements together with a composition operation satisfying four fundamental properties: (1) associativity, (2) transitivity, (3) existence of an identity element, and (4) existence of a unique inverse for each element.
- We then naturally explore subgroups, which are simpler versions of projective transformations.

Euclidean and Similarity Transformations: Motivation

- Changes in coordinate systems in the world are generally Euclidean transformations.
- When the image plane of a camera is parallel to a planar surface and when the camera is moved parallel to or perpendicular to the surface, the image geometry of objects on the surface is changed by a similarity transformation.
- Metric properties are retained by similarity transformations.

Euclidean/Similarity Transformation Matrix Structure

$$\begin{pmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{pmatrix} \tag{2}$$

or, more compactly,

$$\begin{pmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{pmatrix} \tag{3}$$

- There are four degrees of freedom to this transformation.
- Rotation, scaling and translation are combined into a single matrix. This shows another advantage of using homogeneous coordinates.
- The scale factor s must be positive for this to be a similarity transformation. If $s = 1$, it is a Euclidean transformation.
- Angles between vectors are preserved by this transformation.
- Lengths are scaled by s.

Affine Transformations: Motivation

- Affine transformations are intermediate between similarity and projective transformations (in the group hierarchy).
- The motion of a planar surface can often be modeled to good accuracy using an affine transformation.

Affine Transformation Matrix Structure

$$\begin{pmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{pmatrix} \tag{4}$$

or, more compactly,

$$\begin{pmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{pmatrix} \tag{5}$$

- $\mathbf{A}$ must be full rank.
- $\mathbf{t}$ is the translation.

Affine Transformation Properties

- There are six degrees to the affine transformation; the full projective transformation has eight.
- Angles and lengths are not preserved.
- Parallel lines remain parallel.
- The ratio of lengths of parallel segments remains fixed.
- The ratio of two areas remains fixed.

Affine Transformation Decomposition

The 2×2 affine matrix $\mathbf{A}$ may be factored as

$$\mathbf{A} = \mathbf{R}(\theta)\mathbf{R}(-\phi) \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \mathbf{R}(\phi). \tag{6}$$

- The notation $\mathbf{R}(\theta)$ means a 2×2 rotation matrix:

$$\mathbf{R}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{7}$$

- The decomposition is easily seen using the SVD (singular value decomposition).
- Both s_x and s_y are non-zero.

The Line at Infinity

- Recall: The set of all points at infinity in $\mathcal{P}^2$ forms a line whose parameters are

$$\mathbf{l}_\infty = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{8}$$

  This is the *line at infinity*. The associated *points at infinity* that are on this line are $\mathbf{x}_\infty = (x_1, x_2, 0)^\top$.

- An affine transformations moves points at infinity, but leaves them at infinity. In other words, the line at infinity is fixed under affine transformations.
- Algebraically, this is because the last row of the affine transformation matrix is $(0, 0, 1)$.

- A transformation is no longer affine when it moves the line at infinity.
- We will look at an example of this in class.

Decomposition of a Projective Transformation Matrix

- A projective transformation matrix $\mathbf{H}$ may be uniquely decomposed as

$$\mathbf{H} = \mathbf{H}_S \mathbf{H}_A \mathbf{H}_P = \begin{pmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{pmatrix} \begin{pmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{v}^T & v \end{pmatrix} = \begin{pmatrix} \mathbf{A} & v\mathbf{t} \\ \mathbf{v}^T & v \end{pmatrix} \quad (9)$$

  where

  - $s > 0$,
  - $\mathbf{v}$ has two components, v is a scalar.
  - $\mathbf{R}$ is a rotation matrix,
  - $\mathbf{K}$ is upper triangular with $\det(\mathbf{K}) = 1$, and
  - $\mathbf{A} = s\mathbf{R}\mathbf{K} + \mathbf{t}\mathbf{v}^T$.

- Observe how the similarity, affine and projective effects are cleanly separated.
- The vector $\mathbf{v}$ determines the position of the line at infinity.
- Starting from $\mathbf{H}$, the individual matrices and vectors of the factorization are easily computed using a variety of decomposition (factorization) technqies we've already learned.

Removing Projective Distortions of Planar Surfaces

- We can't relate measurements in images to physical reality unless we can undo projective distortion.
- Understanding this leads to deeper understanding of projective transformations.
  - Techniques for building synthetically generated viewpoints, for inserting ads in sporting events, and for marking "first-down" lines in televised American football are based on such an understanding.
- We'll start by moving from projective to affine and then briefly discuss moving from affine to similarity, which is a lot harder.
- Much of this is based on the decomposition in equation 9:

$$\mathbf{H} = \mathbf{H}_S \mathbf{H}_A \mathbf{H}_P = \begin{pmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{pmatrix} \begin{pmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ n\mathbf{v}^T & v \end{pmatrix} = \begin{pmatrix} \mathbf{A} & v\mathbf{t} \\ \mathbf{v}^T & v \end{pmatrix} \quad (10)$$

  - Undoing projective effects requires finding the line at infinity and moving it back to infinity.
  - Undoing affine effects requires finding $\mathbf{K}$, which equates to measuring angles in an affine image.

Projective to Affine

- Recall the following important facts:
  - On a planar surface in the world (in other words prior to projective distortions) parallel lines intersect on the line at infinity.
  - Projections of parallel lines meet at vanishing points.
  - Projective transformations preserve colinearity.
- Therefore, we need to
  - Identify the image location of two vanishing points. Denote these points as $\mathbf{y}_1$ and $\mathbf{y}_2$.
  - Construct the line joining them. This is the image projection of the line at infinity.
  - Find a simple projective transformation that moves this line back to infinity.
- The result is that we've recovered the structure up to an affine transformation: parallel lines are now parallel.
- Undoing affine effects is a bit harder.

Affine to Similarity: Solution Overview

We have already seen a partial solution with the computation of a projective transformation that maps imaged points on the corners of a rectangle to a rectangle. The following discussion generalizes this when there are only lines and not corners.

- The "circular points" are fixed points of a rigid transformation. Of course we can not "see them", but we can measure their effects.
- To do so, we need to know something about conics formed from lines and dual conics formed from points.
- We form the dual absolute conic from the circular points and note that the affine transformation of this conic has a particularly simple form, just involving the matrix $\mathbf{K}$ from (9).
- We then form constraints on this dual absolute conic from the image angles between lines that are supposed to be perpendicular in the world. It turns out that knowing two sets of perpendicular lines is enough to fully constrain the matrix $\mathbf{K}$.

We will discuss some of the details, especially the circular points and line conics. Most of the rest will be left uncovered and you will not be responsible for it.

Circular Points

- Consider the similarity matrix

$$\mathbf{H}_S = \begin{pmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{pmatrix} \tag{11}$$

- The eigenvalues of this matrix are $1, e^{i\theta}, e^{-i\theta}$.
- The eigenvectors corresponding to the last two eigenvalues are

$$\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \tag{12}$$

  These are the *circular points*.
- All circles include these points, as we can easily see.
- All similarity transformations preserve the circular points.
- This suggests that if we can identify the locations of the circular points in the image coordinate system, then we can find a transformation moving them back to their proper location. This would remove affine effects.
- Unfortunately, we can't measure their locations in any way because they are complex. Instead, we measure them indirectly through their dual conic.

Preliminary: Conic Formed from Two Lines, Dual Conics Formed from Two Points

- Consider two lines in $\mathcal{P}^2$, with coordinate vectors $\mathbf{l}$ and $\mathbf{m}$.
- We can express a quadratic form that is satisfied for only points $\mathbf{x}$ on one or both of these lines:

$$\mathbf{C}_{lm} = \mathbf{l}\mathbf{m}^T + \mathbf{m}\mathbf{l}^T. \tag{13}$$

- It is straightforward to show that $\mathbf{x}^T\mathbf{C}_{lm}\mathbf{x} = 0$ if and only if $\mathbf{x}$ is on line $\mathbf{l}$ or $\mathbf{m}$.
- $\mathbf{C}_{lm}$ is a conic, but of rank 2, a seemingly strange best, but an important one.
- As an example, we will form a conic from the x axis and the line $x = y$.
- Analogously, from two points $\mathbf{x}$ and $\mathbf{y}$ we can form a line (dual) conic:

$$\mathbf{C}^*_{xy} = \mathbf{x}\mathbf{y}^T + \mathbf{y}\mathbf{x}^T. \tag{14}$$

  We can show that a line is on this dual conic if it contains one or both of these points.

$\mathbf{C}_\infty^*$ — The Conic Dual to the Circular Points

- We can form the conic dual to the circular points. The result is

$$\mathbf{C}_\infty^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{15}$$

  Note: we are doing this in a coordinate system that has not been projectively or affinely distorted

- Let's look at projective transformations of this conic. Start by reversing the order of the decomposition in equation 9:

$$\mathbf{H} = \mathbf{H}_P \mathbf{H}_A \mathbf{H}_S = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{v}^T & v \end{pmatrix} \begin{pmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{pmatrix} \begin{pmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{A}' & \mathbf{t}' \\ \mathbf{v}'^T & v' \end{pmatrix} \tag{16}$$

  where $\mathbf{A}' = s\mathbf{KR}$, $\mathbf{t}' = \mathbf{Kt}$, $\mathbf{v}'^T = s\mathbf{v}^T\mathbf{KR}$, and $v' = \mathbf{v}^T\mathbf{Kt} + v$.

  - This decomposition, like the one above, is unique. The terms are different (and a bit more complicated).

- Using this, the effect of a projective transformation on $\mathbf{C}_\infty^*$ is

$$\begin{aligned} \mathbf{C}_\infty^*{}' &= \mathbf{H}\mathbf{C}_\infty^*\mathbf{H}^T \\ &= \mathbf{H}_P\mathbf{H}_A\mathbf{H}_S\mathbf{C}_\infty^*\mathbf{H}_S^T\mathbf{H}_A^T\mathbf{H}_P^T \\ &= \mathbf{H}_P\mathbf{H}_A\mathbf{C}_\infty^*\mathbf{H}_A^T\mathbf{H}_P^T \\ &= \begin{pmatrix} \mathbf{KK}^T & \mathbf{KK}^T\mathbf{v} \\ \mathbf{v}^T\mathbf{KK}^T & \mathbf{v}^T\mathbf{KK}^T\mathbf{v} \end{pmatrix}. \end{aligned} \tag{17}$$

  Note that $\mathbf{K}$ is 2×2, and $\mathbf{v}$ is 2×1.

- This matrix has only 4 degrees of freedom because it symmetric, unique only up to scale, and rank 2.

- If there are no projective distortions, then the matrix reduces to

$$\mathbf{C}_\infty^*{}' = \begin{pmatrix} \mathbf{KK}^T & \mathbf{0} \\ \mathbf{0}^T & 0 \end{pmatrix}. \tag{18}$$

  which has only two degrees of freedom.

Angles and $\mathbf{C}_\infty^*{}'$

- Let $\mathbf{l} = \begin{pmatrix} l_1 & l_2 & l_3 \end{pmatrix}^T$ and $\mathbf{m} = \begin{pmatrix} m_1 & m_2 & m_3 \end{pmatrix}^T$ be two lines, and suppose $\mathbf{C}_\infty^*{}'$ has not been distorted by either an affine or projective transformation.

- We can measure the angle between these two lines with respect to the dual absolute conic:

$$\cos\theta_{lm} = \frac{\mathbf{l}^T \mathbf{C}^*_\infty \mathbf{m}}{(\mathbf{l}^T \mathbf{C}^*_\infty \mathbf{l})^{1/2} (\mathbf{m}^T \mathbf{C}^*_\infty \mathbf{m})^{1/2}} = \frac{l_1 m_1 + l_2 m_2}{(l_1^2 + l_2^2)^{1/2} + (m_1^2 + m_2^2)^{1/2}} \tag{19}$$

- If we apply a projective transformation to the lines, we can see that this relation still holds:

$$\cos\theta_{lm} = \frac{\mathbf{l}'^T \mathbf{C}^*_\infty{}' \mathbf{m}'}{(\mathbf{l}'^T \mathbf{C}^*_\infty \mathbf{l}')^{1/2} (\mathbf{m}'^T \mathbf{C}^*_\infty \mathbf{m}')^{1/2}}. \tag{20}$$

- Here, $\mathbf{C}^*_\infty{}'$ is the parameter matrix of the dual absolute conic after the transformation, and $\mathbf{l}'$ and $\mathbf{m}'$ are the tranformed vectors of line parameters. The angle $\cos\theta_{lm}$ is the angle in the original, undistorted coordinate system.

Undoing Affine Distortions — Metric Rectification

- We still can't "see" the dual absolute conic or the circular points in the image. We can generate constraints on them, however, and therefore infer their parameters!

- Here's one way. Suppose $\mathbf{l}_1$ and $\mathbf{m}_1$ and $\mathbf{l}_2$ and $\mathbf{m}_2$ are two sets of perpendicular lines in the undistorted image. Then, following an affine transformation (remember, we undid the projective transformation by finding the line at infinity),

$$\mathbf{l}_1'^T \mathbf{C}^*_\infty{}' \mathbf{m}_1' = 0 \tag{21}$$

  and

$$\mathbf{l}_2'^T \mathbf{C}^*_\infty{}' \mathbf{m}_2' = 0, \tag{22}$$

  where

$$\mathbf{C}^*_\infty{}' = \begin{pmatrix} \mathbf{K}\mathbf{K}^T & \mathbf{0} \\ \mathbf{0}^T & 0 \end{pmatrix}, \tag{23}$$

  as in 18.

- These reduce to

$$\begin{pmatrix} l_{11}' & l_{12}' \end{pmatrix} \mathbf{S} \begin{pmatrix} m_{11}' \\ m_{12}' \end{pmatrix} \tag{24}$$

  and

$$\begin{pmatrix} l_{21}' & l_{22}' \end{pmatrix} \mathbf{S} \begin{pmatrix} m_{21}' \\ m_{22}' \end{pmatrix} \tag{25}$$

  where $\mathbf{S} = \mathbf{K}\mathbf{K}^T$

- $\mathbf{S}$ is clearly symmetric, is positive definite, and has unit determinant.

- This gives enough information to solve for $\mathbf{S}$.
- We can therefore solve for $\mathbf{K}$, which gives us the affine transformation matrix.

Summary

- Projective transformations: 3×3 invertible matrix.
- Projective transformations model image formation for planar surfaces.
- Hierarchy of transformations: Euclidean, similarity, affine and projective.
- A projective transformation may be uniquely decomposed into similarity, affine, and purely projective transformations.
- "Stratified reconstruction" is used to undo the effects of projective and affine transformations and therefore recover the original, similiarity structure in an image:
  - Finding the projection of the line at infinity is used to undo projective effects.
  - The transformation of know angles is used to find the absolute dual conic and from there the affine component of the transformation.
- Here's a parting thought on the power of projective transformations: circles and ellipses are equivalent under affine transformations; circles, ellipses and hyperbolas are equivalent under projective transformations!

Practice Problems and Potential Test Questions

1. Prove that for projective transformation matrix $\mathbf{H}$, line conics transform as $\mathbf{C}^{*\prime} = \mathbf{H}\mathbf{C}^{*}\mathbf{H}^{T}$.
2. Show that similarity transformations satisfy the properties of a group.
3. Find the line conic that is dual to the points $(0, 0, 1)$ and $(1, 1, 1)$. What is the nullspace of this conic? Give its geometric meaning.
4. Suppose we have computed a transformation matrix $\mathbf{H}$ and we want to apply this transformation to an image, I. Sketch a procedure for doing so. What problems might you run into along the way? Is there any advantage to using the inverse mapping $\mathbf{H}^{-1}$?
5. Prove that any circle contains the circular points. Do this by first forming the general conic matrix for a circle.
6. Prove that the decomposition in equation 16 is unique if $s > 0$ and $\det \mathbf{K} = 1$.
7. Find the inverse of a similarity transformation as a function of s, $\mathbf{R}$ and $\mathbf{t}$. Give a geometric interpretation in terms of translation, rotation and scaling.
8. As described above, the 2×2 affine matrix $\mathbf{A}$ may be factored as

   $$\mathbf{A} = \mathbf{R}(\theta)\mathbf{R}(-\phi)\begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix}\mathbf{R}(\phi).$$

   What happens to the affine transformation if $s_x = s_y$?

Homework 6 Problems

Solutions to the following problems are due at the start of class on Monday, March 6, 2006.

1. (**15 points**)
   (a) Given n points $\mathbf{x}_i$ in $\mathcal{P}^2$, find necessary and sufficient conditions to show that these points are collinear.
   (b) Given n lines $\mathbf{l}_i$ in $\mathcal{P}^2$, find necessary and sufficient conditions to show that these lines intersect at a single point.
   (c) Show that these two problems are "duals" of each other in the sense that if you interchange the roles of lines and points in the proofs, the proofs are identical.
2. (**10 points**) How many intersection points are there between a line and a conic? Find them. To do this, let $\mathbf{x}$ and $\mathbf{y}$ be two points on the line and let $\mathbf{C}$ be the conic matrix. Write the line parametrically, and solve from there.