CSci 6974 and ECSE 6966 Math. Tech. for Vision, Graphics and Robotics
Lecture 12, March 2, 2006
Projective Geometry in Three Dimensions

Overview

These notes follow the same general outline as the Lecture 10 and 11 notes.

1. Points, lines, planes and quadrics
2. Transformations
3. Absolute conic and dual absolute quadric
4. Projective equivalences between quadrics (conics)

Points and Planes in $\mathcal{P}^3$

- A point is specified up to a scale factor using a 4-coordinate vector, $\mathbf{x} = (x_1, x_2, x_3, x_4)^\top$, with at least one x_i being non-zero.
- A plane is specified up to a scale factor using a 4-coordinate vector, $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4)^\top$, with at least one π_i being non-zero.
- Point $\mathbf{x}$ is on plane $\boldsymbol{\pi}$ if and only if
$$\boldsymbol{\pi}^\top \mathbf{X} = 0. \tag{1}$$
- Three points $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ determine a plane as the null space of the 3×4 matrix
$$\begin{pmatrix} \mathbf{x}_1^\top \\ \mathbf{x}_2^\top \\ \mathbf{x}_3^\top \end{pmatrix} \tag{2}$$
Under what conditions is this plane not unique?
- Three planes determine a point, except when all three planes intersect along a line.

Lines in $\mathcal{P}^3$

Lines have four degrees of freedom in $\mathcal{P}^3$ and can not be represented as cleanly as points or planes.

- The simplest representation is parametric. Given two points, $\mathbf{x}_1$ and $\mathbf{x}_2$, any other point on the line can be described as a linear combination:

$$\mathbf{x}(t) = t\mathbf{x}_1 + (1-t)\mathbf{x}_2. \tag{3}$$

  This is particularly nice when one of the points is the point at infinity on the line.

- Null-space and span: Given two points $\mathbf{x}_1$ and $\mathbf{x}_2$ form the 2×4 matrix

$$\mathbf{W} = \begin{pmatrix} \mathbf{x}_1^\top \\ \mathbf{x}_2^\top \end{pmatrix}. \tag{4}$$

  Any point in the row space of $\mathbf{W}$ is on the line, and any plane containing the line is in the (two-dimensional) null space of $\mathbf{W}$. Dual properties can be constructed for the matrix

$$\mathbf{W}^* = \begin{pmatrix} \boldsymbol{\pi}_1^\top \\ \boldsymbol{\pi}_2^\top \end{pmatrix}. \tag{5}$$

  formed from two planes.

- Plücker matrices: A 4×4 skew-symmetric matrix:

$$\mathbf{L} = \mathbf{x}_1\mathbf{x}_2^\top - \mathbf{x}_2\mathbf{x}_1^\top. \tag{6}$$

  Note that $\mathbf{L}$ is rank 2. Any plane containing the line is in the nullspace of $\mathbf{L}$.

- Plücker coordinates: See Hartley & Zisserman, Chapter 2.

Projective Geometry of a Plane in $\mathcal{P}^3$

- The plane parameters $\boldsymbol{\pi}$ determine a set of points that forms a two dimensional projective space. We can think of there being a two-dimensional coordinate system on this plane. The points on the plane also sit in a three-dimensional projective space.

- If $\mathbf{u}$ represents a point in the 2d coordinate system and $\mathbf{x}$ represents the same point in $\mathcal{P}^3$, we would like a mapping between $\mathbf{u}$ and $\mathbf{x}$.

- We need to write a mapping

$$\mathbf{x} = \mathbf{M}\mathbf{u} \tag{7}$$

  where $\mathbf{M}$ is 4×3.

- Since $\boldsymbol{\pi}^\top\mathbf{x} = 0$, we have

$$\boldsymbol{\pi}^\top\mathbf{M}\mathbf{x} = 0 \tag{8}$$

  Since this must be true of all $\mathbf{x}$, we must have

$$\boldsymbol{\pi}^\top\mathbf{M} = \mathbf{0} \tag{9}$$

  which means $\mathbf{M}$ sits in the (three-dimensional) nullspace of $\boldsymbol{\pi}^\top$.

– This is a different way of thinking about a nullspace — a matrix as a nullspace of a vector, but it is perfectly legitimate. It is important because $\boldsymbol{\pi}$ is given and $\mathbf{M}$ is what we are trying to discover.

- One possible $\mathbf{M}$ is
$$\mathbf{M} = \begin{pmatrix} \pi_4 & 0 & 0 \\ 0 & \pi_4 & 0 \\ 0 & 0 & \pi_4 \\ -\pi_1 & -\pi_2 & -\pi_3 \end{pmatrix} \tag{10}$$

- Finally, a 2d projective transformation $\mathbf{H}$ on $\mathbf{x}$, induces the inverse transformation on $\mathbf{M}$:
$$\mathbf{M}' = \mathbf{M}\mathbf{H}^{-1}. \tag{11}$$

Quadrics and Dual Quadrics

- The 3d analog of a conic is a quadric:
$$\mathbf{x}^\top \mathbf{Q}\mathbf{x} = 0. \tag{12}$$
where $\mathbf{Q}$ is 4×4 and symmetric. $\mathbf{Q}$ therefore has 9 degrees of freedom.

- The notion of a dual quadric applies to planes. If $\mathbf{Q}^*$ is the adjoint (cofactor) matrix of $\mathbf{Q}$, and $\boldsymbol{\pi}^\top \mathbf{x} = 0$, then
$$\boldsymbol{\pi}^\top \mathbf{Q}^* \boldsymbol{\pi} = 0 \tag{13}$$
if and only if
$$\mathbf{x}^\top \mathbf{Q}\mathbf{x} = 0. \tag{14}$$

Projective Transformations

- Projective transformations preserve coplanarity of points and are described by 4×4, homogeneous, rank-4 matrix $\mathbf{H}$:
$$\mathbf{x}' = \mathbf{H}\mathbf{x}. \tag{15}$$
where here $=$ is "up to a scale factor".

- Transformation of planes, quadrics and dual quadrics is analogous to transformation of lines, conics and dual conics is $\mathcal{P}^2$.

Hierarchy of Transformations

The subgroup hierarchy of transformations is analogous to $\mathcal{P}^2$:

- Similarity transformations have 7 degrees of freedom:
$$\mathbf{H}_A = \begin{pmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{pmatrix} \tag{16}$$
where $\mathbf{R}$ is orthogonal.

- Affine transformations have 12 degrees of freedom:

$$\mathbf{H}_A = \begin{pmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{pmatrix} \tag{17}$$

  where $\mathbf{A}$ is rank 3.

- Projective transformations have 15 degrees of freedom:

$$\mathbf{H}_P = \begin{pmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^\top & v \end{pmatrix} \tag{18}$$

- The sole difference between affine and projective is in the 4th row.

Points at Infinity and the Plane at Infinity

- These are points of the form $\mathbf{x} = (x_1, x_2, x_3, 0)^\top$.
- Parameters of the plane at infinity are $\boldsymbol{\pi}_\infty = (0, 0, 0, 1)^\top$.
- Lines intersect the plane at infinity at a single point — the direction of the line.
- Planes intersect the plane at infinity along an entire line. This line contains the vanishing points for all directions on the plane.
- The plane at infinity is fixed as a set (points can move around on the plane) under a transformation $\mathbf{H}$ if and only if $\mathbf{H}$ is affine.

The Absolute Conic

- This is a **conic** on $\boldsymbol{\pi}_\infty$. Points in $\mathcal{P}^3$ on this conic have the equations

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 &= 0 \\ x_4 &= 0 \end{aligned}$$

- This conic, denoted $\boldsymbol{\Omega}_\infty$, has no real points on it!
- Its relevance can be seen by considering line directions, $\mathbf{d}_1$ and $\mathbf{d}_2$. These are 3-component vectors. (Their 4th component would be 0.)
- The angle between these directions is:

$$\cos\theta = \frac{\mathbf{d}_1^\top \boldsymbol{\Omega}_\infty \mathbf{d}_2}{\sqrt{(\mathbf{d}_1^\top \boldsymbol{\Omega}_\infty \mathbf{d}_1)\,(\mathbf{d}_2^\top \boldsymbol{\Omega}_\infty \mathbf{d}_2)}} \tag{19}$$

- This relationship is preserved under projective transformations that move the absolute conic.

The Absolute Dual Quadric

- The dual of $\mathbf{\Omega}_\infty$ is a rank-3 quadric called the dual absolute quadric:

$$\mathbf{Q}_\infty^* = \begin{pmatrix} \mathbf{I}_{3\times 3} & \mathbf{0} \\ \mathbf{0}^\top & 0 \end{pmatrix}. \tag{20}$$

- $\mathbf{H}$ fixes $\mathbf{Q}_\infty^*$ if and only if $\mathbf{H}$ is a similarity transformation.
- The plane at infinity is the null vector of $\mathbf{Q}_\infty^*$.
- These two facts imply that knowing where a projective transformation has moved $\mathbf{Q}_\infty^*$ is enough to remove projective and affine distortions.
- The angle between planes (normal vectors) is determined by the dual absolute conic:

$$\cos\theta = \frac{\boldsymbol{\pi}_1^\top \mathbf{Q}_\infty^* \boldsymbol{\pi}_2}{\sqrt{(\boldsymbol{\pi}_1^\top \mathbf{Q}_\infty^* \boldsymbol{\pi}_1)\,(\boldsymbol{\pi}_2^\top \mathbf{Q}_\infty^* \boldsymbol{\pi}_2)}}. \tag{21}$$

Projective Equivalences Between Quadrics

- We can learn more about the power of projective transformations by studying their effects on conics and quadrics.
- Let $\mathbf{Q}$ be a full-rank quadric matrix (conic in 2d).
- Apply the following operations:
  1. Compute the spectral decomposition of $\mathbf{Q}$:

  $$\mathbf{Q} = \mathbf{U}\mathbf{W}\mathbf{U}^\top. \tag{22}$$

  2. Apply the projective transformation $\mathbf{H}_1 = \mathbf{U}$ to $\mathbf{Q}$, leaving

  $$\mathbf{Q}^1 = \mathbf{W} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} \tag{23}$$

  This centers and orients the quadric along the axes.
  3. Apply the transformation:

  $$\mathbf{H}_2 = \begin{pmatrix} \sqrt{|\lambda_1|} & 0 & 0 & 0 \\ 0 & \sqrt{|\lambda_2|} & 0 & 0 \\ 0 & 0 & \sqrt{|\lambda_3|} & 0 \\ 0 & 0 & 0 & \sqrt{|\lambda_4|} \end{pmatrix} \tag{24}$$

  4. This leaves a result $\mathbf{Q}^2$ with ± 1 on the diagonal — it effectively normalizes the axes.

5. Multiply by -1 if there are more -1 than +1.
6. Apply a permutation transformation to order the matrix so that the +1's are on the diagonals before the -1's.

All of the transformations we have applied may be combined into a single projective transformation.

- The result is either 4, 3, or 2 +1's, producing the equations

$$\begin{aligned} x^2 + y^2 + z^2 &= -1 \\ x^2 + y^2 + z^2 &= 1 \\ x^2 + y^2 - z^2 &= 1 \end{aligned} \tag{25}$$

- The first is a quadric containing all imaginary points. The second is a unit sphere; all spheres, ellipsoids, and hyperboloids of one sheet may be mapped to this unit sphere. The third is a hyperboloid of two sheets.
- Using a similar analysis we can show that circles, ellipses, parabolas and hyperbolas in $\mathcal{P}^2$ are all projectively equivalent!
- This shows the power of projective transformations to distort shapes.

Practice / Test Problems

1. Give the general quadric matrix form of a sphere. Find the intersection of this sphere with the plane at infinity.

2. This question explores the relationship between planes in $\mathcal{P}^3$ and the two dimensional projective space on the plane. Given plane $\boldsymbol{\pi} = (1, 0, -1, 0)$. Suppose you want $\mathbf{X}_1 = (0, 0, 0, 1)^\top$ to map to $\mathbf{x}_1 = (0, 0, 1)^\top$ on the plane, $\mathbf{X}_2 = (1, 1, 1, 1)$ to map to $\mathbf{x}_2 = (0, 1, 1)^\top$, and $\mathbf{X}_3 = (1, 0, 1, 1)$ to map to $\mathbf{x}_3 = (1, 0, 1)^\top$. Find the 4×3 mapping matrix $\mathbf{M}$ between points $\mathbf{X}$ on the plane (i.e. points satisfying $\boldsymbol{\pi}^\top \mathbf{X} = 0$) and points $\mathbf{x}$ in $\mathcal{P}^2$.

3. What is the effect of a projective transformation on a Plücker line matrix?

4. Create a projective transformation that maps the plane at infinity to the plane $X_1 = 0$. What does this transformation do to the unit sphere?

5. Prove that any circle intersects the absolute conic in two points.

Homework 6 Problems

1. (**15 points**) Prove that the length of a line segment scales by s under a similarity transformation. Prove that the ratio of the lengths of two colinear line segments is preserved under an affine transformation. Prove that neither of these statements is true under projective transformation.

2. (**15 points**) Suppose you have two images that you know are related by a 2D affine transformation. Suppose you have N corresponding point locations $(x_i, y_i)^\top \leftrightarrow (u_i, v_i)$. What is the minimum number of points required to estimate the parameters of the transformation? Derive the equations for a least-squares estimate of the parameters. Under what conditions on the point positions is this estimate unique?