CSCI 6974 / ECSE 6966

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Diff Geom. of Curves & Surfs.

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Def:
The study of geometric figures using calculus

1 Curves

1-2 Parameterized Curves

Curves are 1D subsets of $\mathbb{R}^3$

Will define curves via functions of real variables

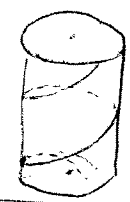
A function is {smooth / differentiable} if at every point derivatives of all orders exist.

$\boxed{\text{DEF}}$: A param. diff. curve is a diff. map $\alpha: I \to \mathbb{R}^3$ of an open interval $I = (a,b)$ of the real line $\mathbb{R}$ into $\mathbb{R}^3$.

Example 1 $\qquad \alpha(t) = (a\cos t,\ a\sin t,\ bt) \quad t \in \mathbb{R}$

a = radius of helix

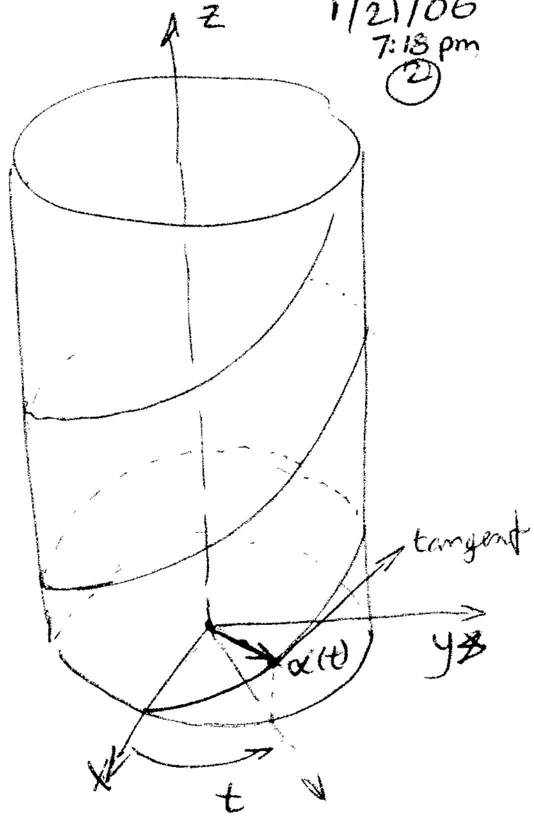
$2\pi b$ = pitch



~~The curve is aka the trace of the map α.~~

~~" " " " the image of I under the map α.~~

$I = (-\infty, \infty)$



$$\alpha' = \frac{d\alpha}{dt} = \left(\frac{dx(t)}{dt}, \frac{dy(t)}{dt}, \frac{dz(t)}{dt} \right)$$

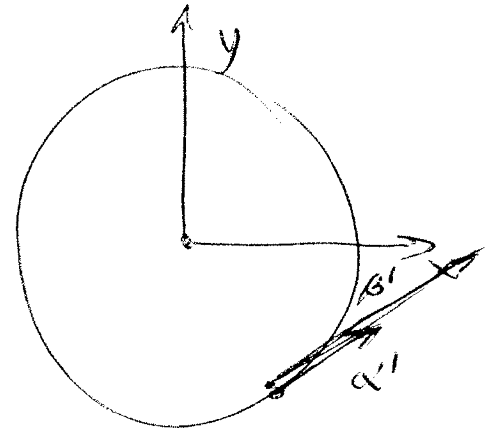
$\alpha' = (x', y', z')$

$\alpha' \equiv$ tangent vector in direction of $\uparrow t$

<u>Example 5</u>. Two distinct parameterized curves

$$\left.\begin{aligned} \alpha(t) &= (\cos t, \sin t) \\ \beta(t) &= (\cos 2t, \sin 2t) \end{aligned}\right\} \text{The trace of both is circle of radius 1}$$

Note $\alpha'(t) = \frac{1}{2}\beta'(t)$



Emphasize:
$\alpha(t)$ & $\beta(t)$ are distinct maps or parameterized curves

$\text{trace}(\alpha(t)) = \text{trace}(\beta(t)) \subset \mathbb{R}^3$

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1-3 Regular Curves - Arc Length

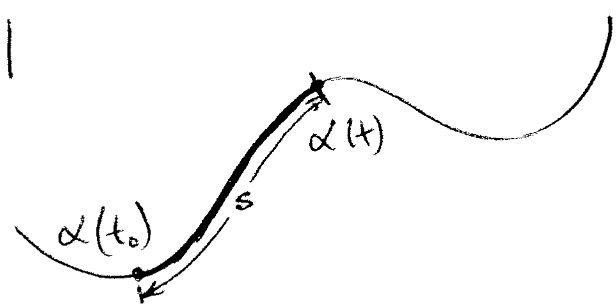
$\boxed{\text{DEF}}$ A parameterized differentiable curve $\alpha : I \to \mathbb{R}^3$ is **regular** if $\alpha'(t) \neq 0$ for any $t \in I$.

From now on consider only param. diff. curves

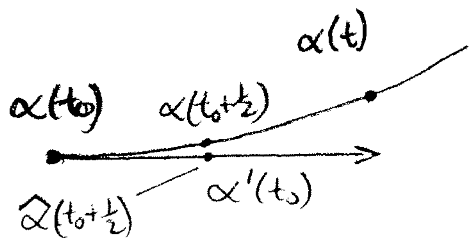
Arc Length $\quad s(t) = \int_{t_0}^{t} |\alpha'(t)|\, dt$

where $|\alpha'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$

$$\frac{ds}{dt} = |\alpha'(t)|$$



How does $\alpha'(t)$ relate to arc length?



Suppose $\Delta t = \frac{1}{2}$

Taylor series: $\quad \alpha(t_0 + \Delta t) = \alpha(t_0) + \alpha' \Delta t + \text{H.O.T.}$

If the parameter t is already arc length,

then $s(t) = \int_{t_0}^{t} dt$

Henceforth we will, w/o loss of generality, consider only differentiable curves parameterized by arc length.

Change of Orientation

Given $\alpha(s) : I \to \mathbb{R}^3$ $\quad I = (a,b)$

~~Then~~ let $\beta(s) = \alpha(-s)$, and $J = (-b,-a)$

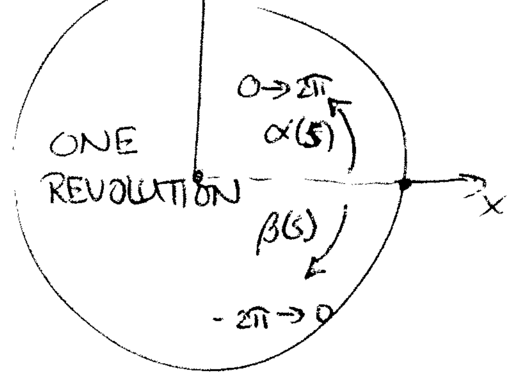
We say $\beta(s)$ is a change of orientation of $\alpha(s)$.

Example: $\alpha = (\cos(s), \sin(s))$

$s \in (0, 2\pi)$

$\beta = (\cos(-s), \sin(-s))$

$s \in (-2\pi, 0)$



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1-4 The Vector Product in $\mathbb{R}^n$

Two ordered bases $e = \{e_i\}$, $f = \{f_i\}$; $i = 1, \dots, n$ of $\mathbb{R}^n$ have the <u>same orientation</u> if the matrix of change of basis has positive determinant.

Basis orientation partitions all bases into 2 equivalence classes: ie they satisfy:

1.) $e \sim e$

2.) if $e \sim f$, then $f \sim e$

3.) if $e \sim f$, $f \sim g$, the $e \sim g$

Let R be the change of basis transformation matrix.

$$ {}^E_F R = \left[{}^E f_1 \;\; {}^E f_2 \;\; {}^E f_3 \right], \qquad \text{Vector } {}^E V = {}^E_F R \, {}^F V $$

Is this correct for bases that are not ortho-normal?

$$ \equiv {}^E f_1 \wedge {}^E f_2 \cdot {}^E f_3 $$

Swapping order changes sign of result.

$$ \equiv \begin{vmatrix} {}^E f_1^T \\ {}^E f_2^T \\ {}^E f_3^T \end{vmatrix} = \begin{vmatrix} {}^E f_1 & {}^E f_2 & {}^E f_3 \end{vmatrix} $$

To be a basis, $|R| \neq 0$.

By changing basis order, ~~[illegible]~~ only sign can change!

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Properties of Vector Product, $\wedge$

Let $u, v \in \mathbb{R}^3$

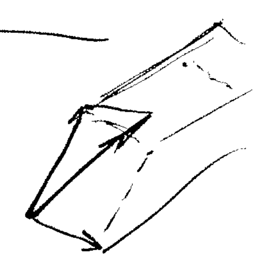
1. Anticommutativity: $u \wedge v = - v \wedge u$
2. Linearity: $(au + bw) \wedge v = a u \wedge v + b w \wedge v$, $a, b \in \mathbb{R}$
3. $u \wedge v = 0$ iff u & v are linearly dependent
4. $(u \wedge v) \cdot u = (u \wedge v) \cdot v = 0$. Follows directly from properties of determinants.
5. $u \wedge v$ is $\perp$ the plane defined by v & u
6. Differentiability: $u(t) \wedge v(t) = \frac{du(t)}{dt} \wedge v(t) + u(t) \wedge \frac{dv(t)}{dt}$

Define right-handed coord. frames to have positive orientation.

Then given $u, v \in \mathbb{R}^3$,

$\{u, v, u \wedge v\}$ is a basis with positive orientation assuming $u \wedge v \neq 0$.

Geometric Interp. of Basis vectors.



$\mathrm{Det}(e_1, e_2, e_3) =$ volume of parallelopiped defined by e_1, e_2, e_3

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Vector product is <u>not</u> associative

$$(u \wedge v) \wedge w \neq u \wedge (v \wedge w)$$

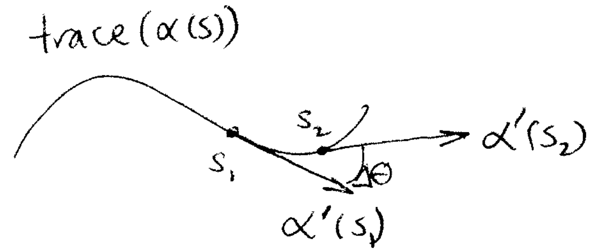
$$= (u \cdot w) v - (v \cdot w) u$$

1-5 The Local Theory of Curves

Parameterized by Arc Length

Let $\alpha : I = (a, b) \longrightarrow \mathbb{R}^3$ be a curve parameterized by arc length s.

Tangent vector $\alpha'(s)$ has unit length



The norm $|\alpha''(s)|$ measures the rate of change of angle of $\alpha'(s)$.

Measures how rapidly the curve "pulls away" from ~~the~~ its tangent at s.

Why use norm?

DEF: $|\alpha''(s)| = k(s)$ is called the curvature of α

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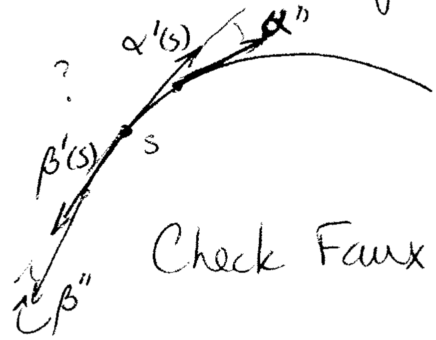
Change of orientation, reverses tangent vector. (9)

if $\beta(-s) = \alpha(s)$

$$\frac{d\beta(-s)}{d(-s)} = -\frac{d\alpha(s)}{ds}$$



Check Faux & Pratt

If α is a straight line, then $k(s) = |\alpha''(s)| = 0$

Claims that $\alpha''(s)$ is invariant under change of orientation

$$\alpha = \begin{bmatrix} s^2+s \\ s \\ 0 \end{bmatrix} \qquad \beta = \begin{bmatrix} s^2-s \\ -s \\ 0 \end{bmatrix}$$

$$\alpha' = \begin{bmatrix} 2s+1 \\ 1 \\ 0 \end{bmatrix} \qquad \beta' = \frac{d\beta}{ds} = \begin{bmatrix} 2s-1 \\ -1 \\ 0 \end{bmatrix} \qquad \frac{d\beta}{d(-s)} = \begin{bmatrix} -2s+ \\ 1 \\ 0 \end{bmatrix}$$

$$\alpha'' = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \qquad \beta'' = \frac{d^2\beta}{ds^2} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \qquad \frac{d^2\beta}{d(-s)^2} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

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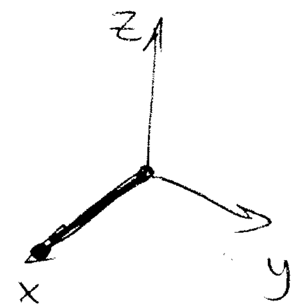
$\cancel{\alpha : 3s+2}$

$$\alpha : (\overset{a}{0}, \overset{b}{1}) \longrightarrow \mathbb{R}^3$$



$$\alpha = \begin{bmatrix} s \\ 0 \\ 0 \end{bmatrix}$$

$$\beta : (\overset{-b}{-1}, \overset{-a}{0}) \longrightarrow \mathbb{R}^3$$

$$\beta = \begin{bmatrix} -s \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{d\alpha}{ds} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \frac{d\beta}{ds} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \qquad \frac{d\beta}{d(s)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

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If $\alpha''(s) \neq 0$ (ie. $k(s) \neq 0$) then

$\alpha''(s) = n(s)\, k(s)$, where $n(s) \perp t(s) = \alpha'(s)$

Note that $\frac{d}{dt}\left(\alpha'(s) \cdot \alpha'(s) = 1\right) \Rightarrow \alpha''(s) \cdot \alpha'(s) = 0$

$\Rightarrow n(s) \perp \alpha'(s) = t(s)$.

The <u>osculating plane</u> is ~~determined~~ ~~by~~ the span of $n(s)$ & $t(s)$

If $k(s) = 0$, the osc. plane is undefined

$\therefore$ DEF: $s \in I$ is a singular pt of order $\underline{1}$ if $\alpha''(s) = 0$

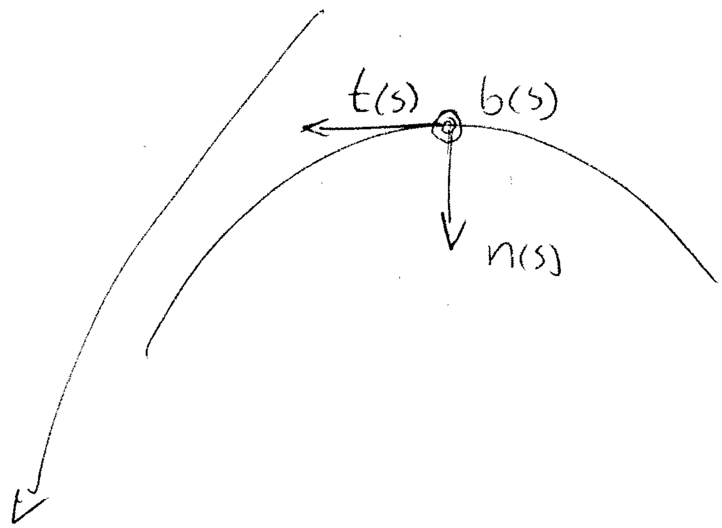
Note if $\alpha'(s) = 0$, s is a sing pt of order 0.

DEF: $b(s) = t(s) \wedge n(s)$ is the normal to the osc. plane. $b(s)$ is called the binormal vector.

Note $b'(s) \perp b(s)$, but also:

$$b'(s) = \underbrace{t'(s) \wedge n(s)}_{0} + t(s) \wedge n'(s) = \underline{t(s) \wedge n'(s)}$$

~~b(s) is a unit vector that measures how quickly the curve pulls away from the osculating plane.~~



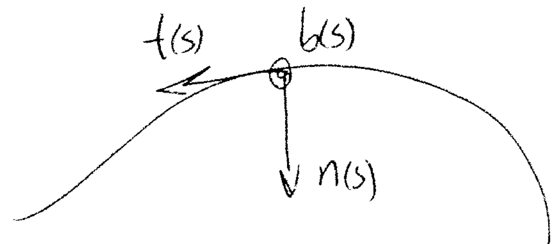
Since $b'(s) = t(s) \wedge n'(s)$, $b'(s) \perp t(s)$.

Since $b'(s) \perp \{b(s), t(s)\}$, $b'(s)$ must be $\parallel n(s)$.

Write $b'(s)$ as $b'(s) = \tau(s)\, n(s)$

DEF: $\tau(s)$ is called the torsion of α and measures how quickly the curve pulls away from the osculating plane at s

DEF: The local coord frame, $\{t(s), n(s), b(s)\}$ is the Frenet trihedron.



We're determined t', b', but not n'

$$n'(s) = b'(s) \wedge t(s) + b(s) \wedge t'(s)$$

$$= -\tau b - kt$$

Thus the deriv of the Frenet trihedron is

$$t' = kn$$

$$n' = -(kt + \tau b)$$

$$b' = \tau n$$

Terminology

tb = rectifying plane

nb = normal plane

line containing n and passing ~~to~~ thru α is called the principal normal

line containing b and passing thru α is called the binormal

The inverse $R = \frac{1}{k}$ is the radius of curvature.

$\boxed{\text{Take a piece of bailing wire}}$

Fundamental Theorem of the Local Theory of Curves

Given differentiable fcns $k(s) > 0$ and $\tau(s)$, $s \in I$, $\exists$ a regular parameterized curve $\alpha : I \to \mathbb{R}^3 \ni s$ is the arc length, $k(s)$ is the curvature, and $\tau(s)$ is the torsion of α.

Moreover, any other curve $\bar{\alpha}$, satisfying the same conditions, differs from α by a rigid motion: that is, $\exists$ an orthogonal linear map ρ of $\mathbb{R}^3$ with positive determinant and a vector $c \ni \bar{\alpha} = \rho \circ \alpha + c$