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Linear Programs

From Luenberger Linear & Nonlinear Programming
Addison Wesley 1984.

Standard Form

$$\boxed{\begin{array}{ll} \underset{x}{\text{Minimize}} & c^T x \\ \text{subject to:} & Ax = b \\ & x \geq 0 \end{array}}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$ are given & constant.
The vector $x \in \mathbb{R}^n$ is the unknown.

$$x \geq 0 \implies x_i \geq 0 \quad \forall \; i \in \{1, \ldots, n\}$$

Other forms convertible to standard form

Slack Variables

$$\begin{array}{ll} \underset{x}{\text{Minimize}} & c^T x \\ \text{s.t.}: & Ax \leq b \\ & x \geq 0 \end{array}$$

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Introduce slack variables, $y \geq 0$, $y \in \mathbb{R}^m$

$$Ax + y = b \;=\; Ax + Iy = b$$

Let $z = \begin{bmatrix} x \\ y \end{bmatrix}$, $D = [A \;\; I]$, $e^T = [c^T \;\; 0]$

Substitution yields:

$$\begin{aligned} \underset{z}{\text{Minimize}} \quad & e^T z \\ \text{s.t.:} \quad & Dz = b \\ & z \geq 0 \end{aligned}$$

Surplus Variables

$$\left.\begin{aligned} \underset{x}{\text{Min}} \quad & c^T x \\ \text{s.t.:} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}\right\} \qquad \begin{aligned} \underset{z}{\text{Min}} \quad & e^T z \\ \text{s.t.:} \quad & Dz = b \\ & z \geq 0 \end{aligned}$$

where $e^T = [c^T \;\; 0]$

$D = [A \;\; -I]$

$z = \begin{bmatrix} x \\ y \end{bmatrix}$

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Free Variables

Suppose x_1 is unconstrained.

Then replace x_1 with $x_1 = x_1^+ - x_1^-$,

$$x_1^+, x_1^- \geq 0$$

Substitute $x_1^+ - x_1^-$

everywhere that x_1 appears.

The resulting LP will have one additional variable.

Suppose all variables are free.

$$\min_x \; c^T x$$
$$\text{s.t. } Ax = b$$

Let $x = x^+ - x^-$, $x^+, x^- \geq 0$

$$\min_{x^+, x^-} \; c^T(x^+ - x^-)$$
$$\text{s.t. } A(x^+ - x^-) = b$$

Finally

$$\min_z \; e^T z$$
$$\text{s.t. } Dz = b$$
$$z \geq 0$$

where $e^T = [c^T \; -c^T]$, $D = [A \; -A]$, $z = [x^+ \; x^-]$

Note: LP increases in size.

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Free Variables (second method)

Suppose x_1 is unconstrained.

Find an equation in $Ax=b$ such that $a_{i1} \neq 0$.

Solve for x_1 in terms of other elements of x.

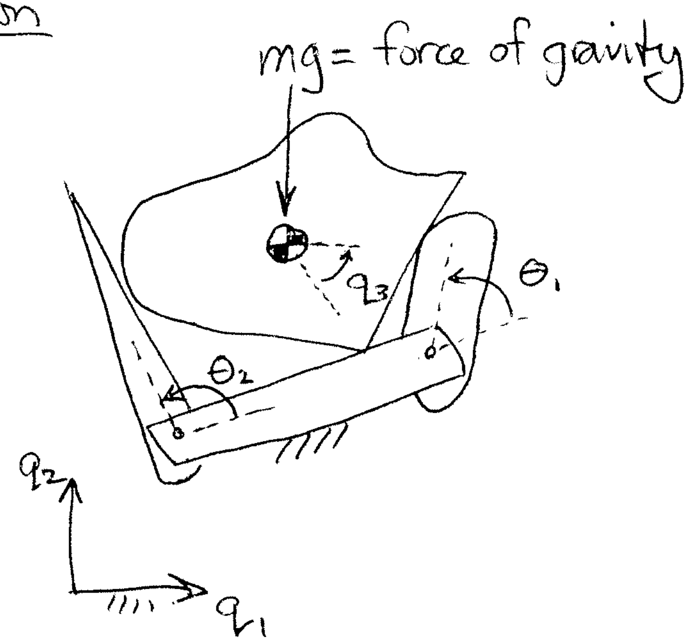
Substitute into LP to eliminate x_1.

LP reduces in size.

Go to Page 4.1 Now

Example: Dexterous Manipulation

Let $q=(q_1,q_2,q_3)$ denote pos. & orient. of grasped object, θ_1, θ_2 denote finger angles.



Assume no friction.

Assume very slow motion.

$\boxed{\text{Problem: given } \delta\theta_1, \delta\theta_2\text{; determine } \delta q.}$

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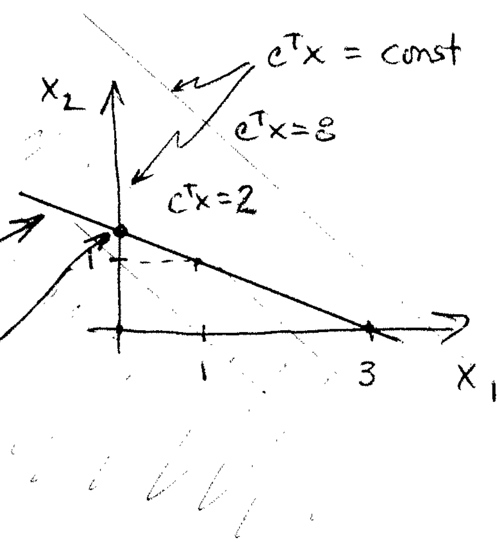
Geometric Interpretation

$$\min_{x} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\text{s.t. } \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3$$

$$x_1, x_2 \geq 0$$



$Ax = b$

optimal solution

$$x^* = \begin{bmatrix} 0 \\ 1.5 \end{bmatrix}$$

$$c^T x^* = 1.5$$

Constraints:

1.) Minimize gravitational potential energy, $mg\,q_2$

or $\quad -\underbrace{\begin{bmatrix} 0 & -mg & 0 \end{bmatrix}}_{g_{ext}} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \Rightarrow \boxed{c^T \delta q}$

2.) Nonpenetration. Distance functions, $\psi(q,\theta) \geq 0$.

Each contact has a distance function.

$$\psi(q+\delta q, \theta+\delta\theta) = \psi(q,\theta) + \frac{\partial\psi}{\partial q}\delta q + \frac{\partial\psi}{\partial\theta}\delta\theta + \dots \geq 0$$

At a contact $\psi_i(q,\theta) = 0$.

Linear approximation:

$$\frac{\partial\psi}{\partial q}\delta q + \frac{\partial\psi}{\partial\theta}\delta\theta \geq 0$$

$$\boxed{W_n^T \delta q \geq J_n \delta\theta}$$

Quasistatic Manipulation of Frictionless Object: Given $\delta\theta$, ...

$$\min_{\delta q} \; c^T \delta q = -g_{ext}^T \delta q$$

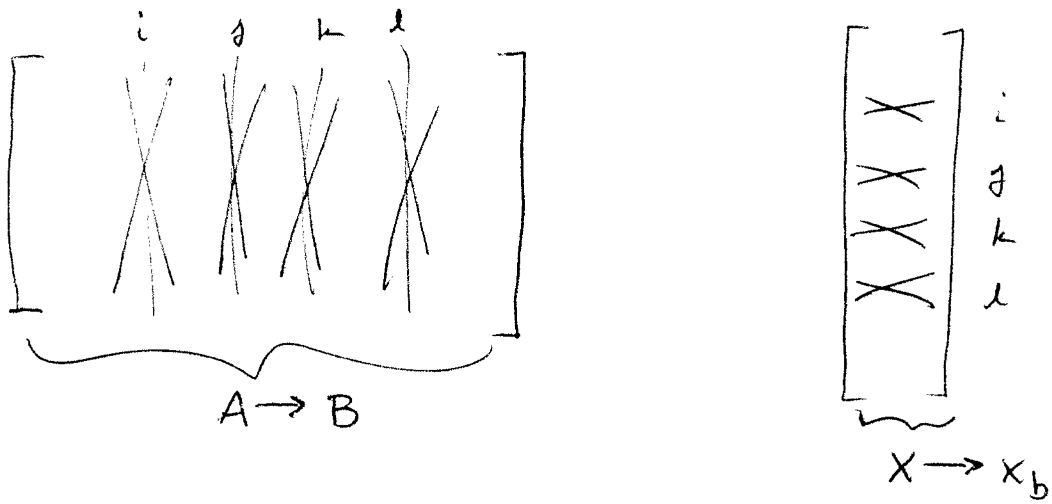
$$\text{s.t.} \quad \underset{\uparrow\, A}{W_n^T}\, \underset{\uparrow\, x}{\delta q} \geq \underset{\uparrow\, b}{\underbrace{J_n \delta\theta}}$$

Note that δq is a free variable.

We could use techniques from previous pages to cast in standard form.

<u>Definition</u>

Basic Solution: Let B be an $m \times m$ nonsingular submatrix formed by deleting columns of A. Then $x_b = B^{-1} b$, $x = (x_b, 0)$ is a <u>basic</u> <u>solution</u>.



x_b are called basic variables.

Note A is $m \times n$. <u>It is assumed that $n > m$</u>

<u>It is assumed that A is full rank</u>

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If A is not full row rank, then either rows of A can be eliminated or no solution exists.

<u>Definition</u>: Feasible Solution: Any vector x satisfying
$Ax = b$, $x \geq 0$ is called a <u>feasible solution</u>.

<u>Def</u>: If x is basic & feasible, then it is called a <u>basic feasible solution</u>.

Return to simple example on page (4.1).

x^* is basic & feasible

$B = [2]$ $\quad$ $x_b = x_2$ $\quad$ $x^* = \begin{bmatrix} 0 \\ \hline 1.5 \end{bmatrix} \leftarrow x_b$

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Fundamental Thm of L.P.

Given an LP in standard form where

A is $(m \times n)$ and of rank m

i.) if there is a feasible solution, there is a basic feasible solution

ii) if there is an optimal feasible sdn, there is an optimal basic feasible solution.

Enumerative Solution Alg. for LP.

Choose m columns of A to form B

$x = (x_b, 0)$

check $x_b \geq 0$

record value of $c^T x$

There are $\binom{n}{m}$ passes required. $\frac{n!}{m!(n-m)!}$

$\boxed{\text{Enumeration is exponential in } n.}$

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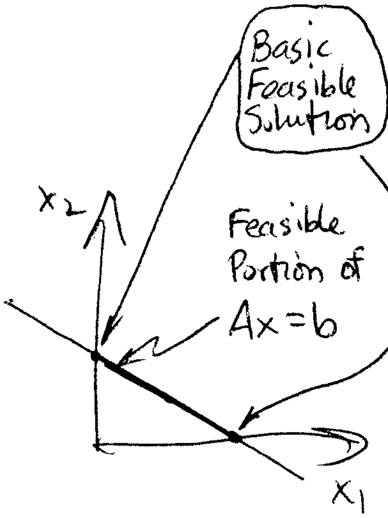
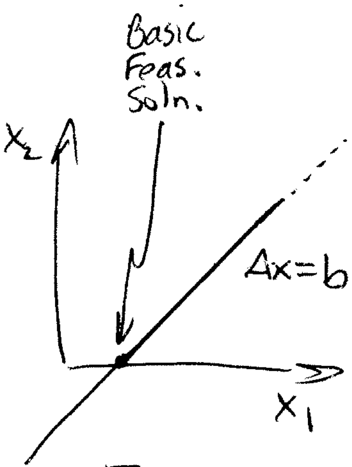
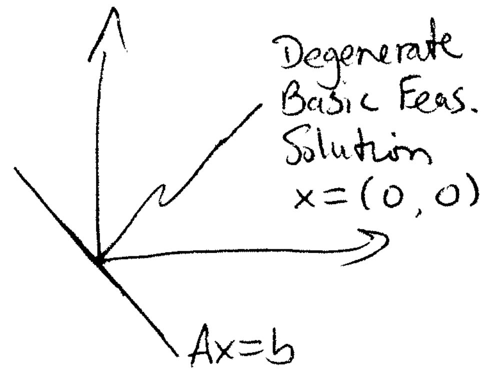


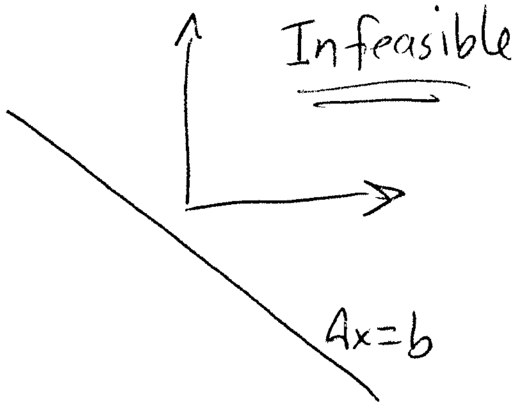
Feasible



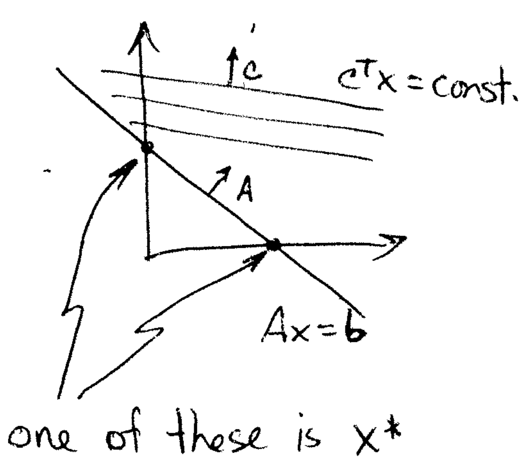
Feasible



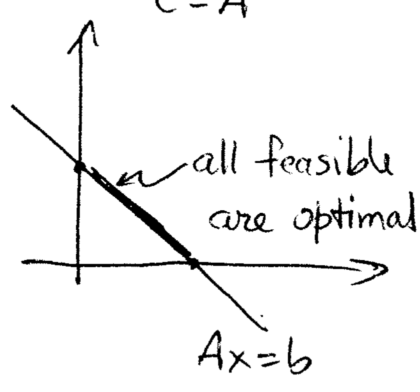
Feasible



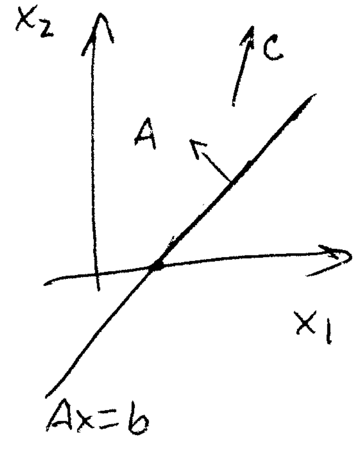
$c \neq A$



$c = A$



x^* unbounded



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Convexity

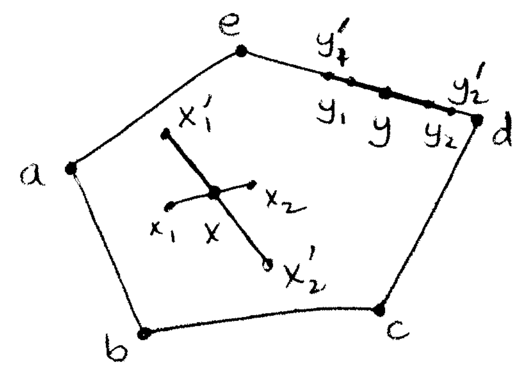
Def: <u>Extreme point</u>: In a convex set C, a point x is an extreme point if there are no two distinct points $x_1 \neq x_2$ in C $\ni$ $x = \alpha x_1 + (1-\alpha) x_2$ for some α, $0 < \alpha < 1$.

Example:

x is <u>not</u> extreme

y is <u>not</u> extreme

a,b,c,d,e are extreme pts.



<u>Theorem</u>: Let A be an $m \times n$ matrix of rank m and b be an element of $\mathbb{R}^m$. Let K be a convex polytope consisting of all n-vectors x satisfying

$$\left.\begin{array}{r} Ax = b \\ x \geq 0 \end{array}\right\} \qquad K = \{x \mid Ax = b,\ x \geq 0\}$$

A vector x is an extreme point of K iff x is a basic feasible solution of $Ax = b$, $x \geq 0$.

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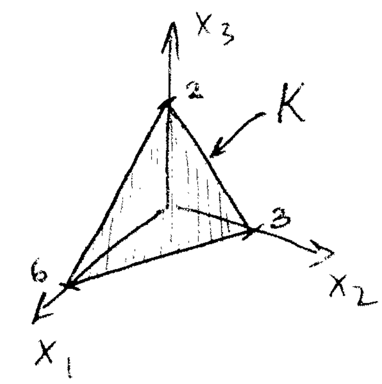
Example

$$A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \qquad b = 6$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\left.\begin{array}{r} Ax = b \\ x \geq 0 \end{array}\right\} = K =$$



How many basic solution?

$$\binom{3}{1} = 3$$

$$B = 1 \Rightarrow x = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$$

$$B = 2 \Rightarrow x = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

$$B = 3 \Rightarrow x = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

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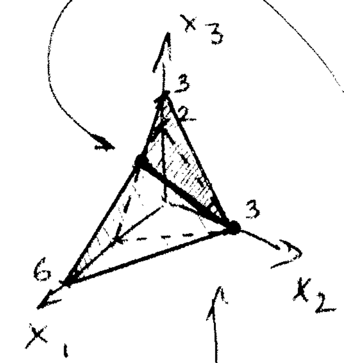
Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$



How many basic feasible solutions?

$\binom{3}{2} = 3$

$\underline{x_3 = 0}$

$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$

$\underline{x_2 = 0}$

$\begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 3 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 1.5 \\ 0 \\ 1.5 \end{bmatrix}$

Duplicate Basic Solutions

$\underline{x_1 = 0}$

$\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 3 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$

Similar example:

$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \xRightarrow[\text{solutions}]{\text{basic}} \left\{ \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}, \cancel{\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}} \right\}$

Not Feasible

Solution Algs for LP's

Dantzig Method

Simplex – search extreme points.

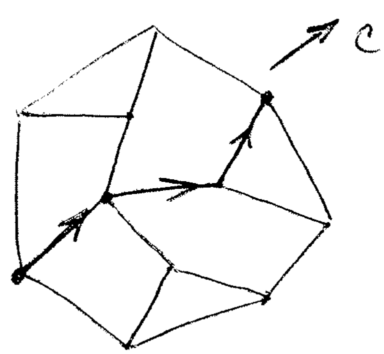
> Worst-case performance is exponential in number of unknowns

Klee & Minty constructed an example with a polytope with 2^n vertices such that the Simplex Alg. visited all vertices before finding a solution. 1972.

> Average-case performance is proportional to the number of constraints ~~unknowns~~ 2m – 3m steps

Graphical view of Simplex Algorithm

Move from vertex to vertex trying to increase $c^T x$



Start

Simplex moves are also known as pivots.

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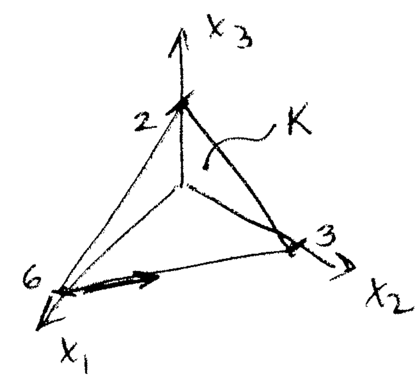
Consider simple problem

$$\text{Min} \quad \begin{bmatrix} 0 & -\tfrac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\text{s.t.} \quad x_1 + 2x_2 + 3x_3 = 6$$



Start with x_1 basic, x_2, x_3 nonbasic

i.e. Solve

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ x_2 &= 0 \\ x_3 &= 0 \end{aligned} \quad \Rightarrow \quad x = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} \quad c^T x = 0$$

Which variable should be basic next?
Equivalently, along which edge of K should we move?

Assume x_2 becomes basic & x_1 becomes nonbasic.

The edge is in the null space of

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ x_3 &= 0 \end{aligned}$$

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parametric eq. of edge in x_1, x_2, plane.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1/2 \\ 0 \end{bmatrix} \alpha = \begin{bmatrix} 6-\alpha \\ 1/2\,\alpha \\ 0 \end{bmatrix} \geq 0 \quad \Rightarrow \quad 0 \leq \alpha \leq 6$$

Suppose x_3 becomes basic

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1/3 \end{bmatrix} \beta = \begin{bmatrix} 6-\beta \\ 0 \\ 1/3\,\beta \end{bmatrix} \geq 0 \quad \Rightarrow \quad 0 \leq \beta \leq 6$$

Which pivot is better?

Consider ~~de~~crease of $c^T x$.

$$\Delta(c^T x) = \begin{bmatrix} 0 & -1/2 & -1 \end{bmatrix} \begin{bmatrix} 6-\alpha \\ 1/2\,\alpha \\ 0 \end{bmatrix} = -1/4\,\alpha \xRightarrow{\text{Biggest Change}} -1.5$$

$$\Delta(c^T x) = \begin{bmatrix} 0 & -1/2 & -1 \end{bmatrix} \begin{bmatrix} 6-\beta \\ 0 \\ 1/3\,\beta \end{bmatrix} = -1/3\,\beta \Rightarrow \boxed{-2.0}$$

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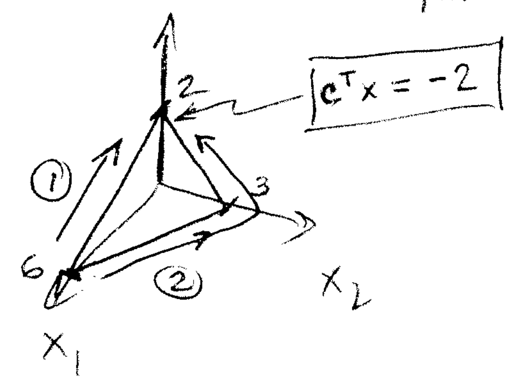
Two paths to optimum

Path 1 is direct

Path 2 takes two steps



How do we know when we've arrived at the global minimum?

Look at 1-D null spaces from current point.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3/2 \\ -1 \end{bmatrix} \alpha \quad , \quad 0 \le \alpha \le 2$$

$$\Delta(c^T x) = \begin{bmatrix} 0 & -1/2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 3/2 \\ -1 \end{bmatrix} \alpha = \frac{1}{4} \alpha$$

Note: Any movement of x along the edge (i.e., $\alpha > 0$) increases $c^T x$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \beta \quad , \quad 0 \le \beta \le 2 \Rightarrow \boxed{\Delta c^T x = \beta}$$

$\beta > 0$ increases $c^T x$.

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Primal / Dual LPs

The soln of an LP generally includes x & λ where λ is the vector of Lagrange multipliers.

Values of Lagrange multipliers indicate how rapidly and in which direction the object function changes due to constraint violation.

PRIMAL	DUAL
Min $c^T x$	Max $b^T \lambda$
s.t. $Ax = b$	s.t. $A^T \lambda \le c$
$x \ge 0$	

Return to Manipulation Problem.

With some algebraic manipulation ...

(Work done by contact forces)

$$\min_{\lambda_n} \; \delta\theta^T J_n^T \lambda_n$$

s.t. $W_n \lambda_n = -g_{ext}$

$\lambda_n \ge 0$

(equilibrium conditions w/ unilateral contacts)

$$\max_{\delta q} \; g_{ext}^T \delta q$$

(Work done by gravity)

s.t. $W_n^T \delta q \ge J_n \delta\theta$

(Nonpenetration)