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①

9:21 pm

## Linear Programs

From Luenberger Linear & Nonlinear Programming

Addison Wesley 1984.

### Standard Form

$$\begin{array}{l} \text{Minimize } c^T x \\ \text{subject to: } Ax = b \\ x \geq 0 \end{array}$$

where  $A \in \mathbb{R}^{m \times n}$ ,  $b \in \mathbb{R}^m$ ,  $c \in \mathbb{R}^n$  are given & constant.

The vector  $x \in \mathbb{R}^n$  is the unknown.

$$x \geq 0 \Rightarrow x_i \geq 0 \quad \forall i \in \{1, \dots, n\}$$

Other forms convertible to standard form

### Slack Variables

$$\begin{array}{l} \text{Minimize } c^T x \\ \text{s.t. : } Ax \leq b \\ x \geq 0 \end{array}$$

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Introduce slack variables,  $y \geq 0$ ,  $y \in \mathbb{R}^m$

$$Ax + y = b = Ax + Iy = b$$

$$\text{Let } z = \begin{bmatrix} x \\ y \end{bmatrix}, \quad D = \begin{bmatrix} A & I \end{bmatrix}, \quad e^T = [c^T \ 0]$$

Substitution yields:

$$\text{Minimize } e^T z$$

$$\text{st.} \cdot Dz = b$$

$$z \geq 0$$

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Surplus Variables

$$\text{Min } c^T x$$

$$\text{st.} \cdot Ax \geq b$$

$$x \geq 0$$

$$\text{Min } e^T z$$

$$\text{st.} \cdot Dz = b$$

$$z \geq 0$$

$$\text{where } e^T = [c^T \ 0]$$

$$D = \begin{bmatrix} A & -I \end{bmatrix}$$

$$z = \begin{bmatrix} x \\ y \end{bmatrix}$$

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### Free Variables

Suppose  $x_1$  is unconstrained.

Then replace  $x_1$  with  $x_1 = x_1^+ - x_1^-$ ,  
 $x_1^+, x_1^- \geq 0$

Substitute  $x_1^+ - x_1^-$

everywhere that  $x_1$  appears.

The resulting LP will have one additional variable.

Suppose all variables are free.

$$\begin{aligned} \text{Min}_x \quad & c^T x \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

$$\text{Let } x = x^+ - x^-, \quad x^+, x^- \geq 0$$

$$\begin{aligned} \text{Min}_{x^+, x^-} \quad & c^T(x^+ - x^-) \\ \text{s.t.} \quad & A(x^+ - x^-) = b \end{aligned}$$

$$\begin{aligned} \text{Finally} \quad & \text{Min}_z \quad e^T z \\ & \text{s.t. } Dz = b \\ & z \geq 0 \end{aligned}$$

$$\text{where } e^T = [c^T \quad -c^T], \quad D = [A \quad -A], \quad z = [x^+ \quad x^-]$$

Note: LP increases in size.

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## Free Variables (second method)

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Suppose  $x_1$  is unconstrained.

Find an equation in  $Ax=b$  such that  $a_{31} \neq 0$ .

Solve for  $x_1$  in terms of other elements of  $x$ .

Substitute into LP to eliminate  $x_1$ .

LP reduces in size.

Go to Page 4.1 Now

Example: Dexterous Manipulation

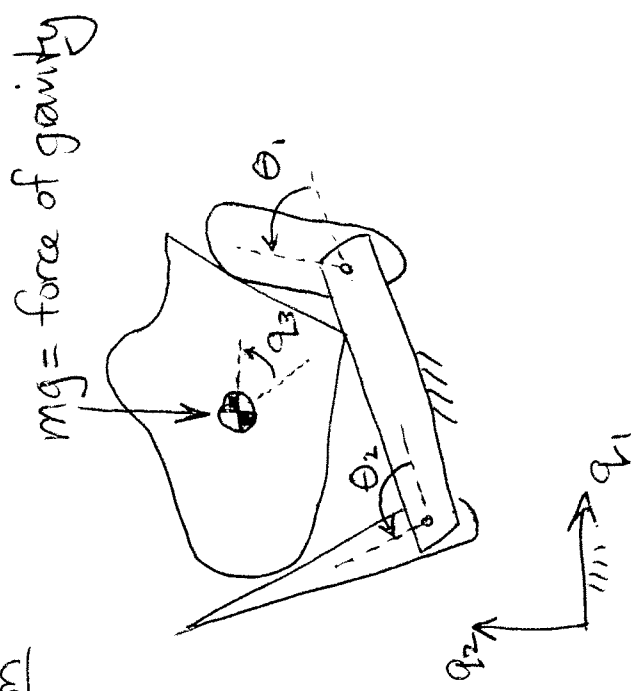
Let  $q = (q_1, q_2, q_3)$

denote pos. & orient.  
of grasped object,

$\theta_1, \theta_2$  denote finger  
angles.

Assume no friction.

Assume very slow motion.



Problem: given  $\delta\theta_1, \delta\theta_2$ ; determine  $\delta q$ .

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(4.1)

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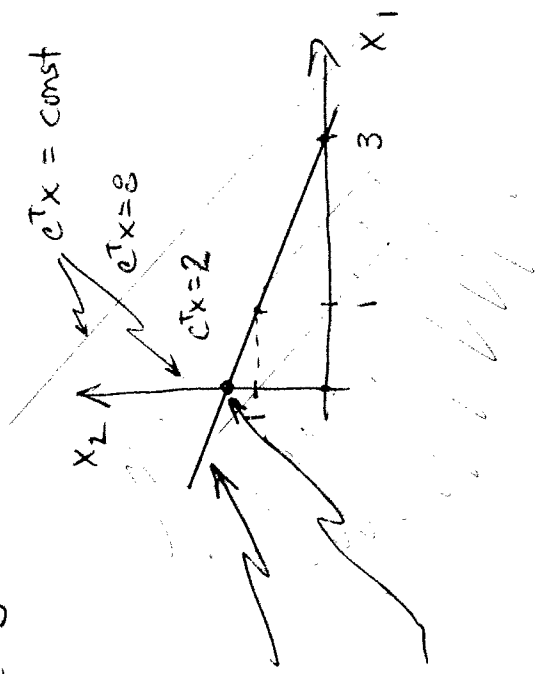
### Geometric Interpretation

$$\text{Min}_x \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\text{s.t.} \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3$$

$$x_1, x_2 \geq 0$$

$$Ax = b$$



optimal  
solution  
 $x^* = \begin{bmatrix} 0 \\ 1.5 \end{bmatrix}$

$$c^T x^* = 1.5$$

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Constraints:

1.) Minimize gravitational potential energy,  $mgq_2$

$$\text{or } \underbrace{-[0 \ -mg \ 0]}_{g_{ext}} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \Rightarrow \boxed{c^T \delta q}$$

2.) Nonpenetration. Distance functions,  $\Psi(q, \theta) \geq 0$ .

Each contact has a distance function.

$$\Psi(q + \delta q, \theta + \delta \theta) = \Psi(q, \theta) + \frac{\partial \Psi}{\partial q} \delta q + \frac{\partial \Psi}{\partial \theta} \delta \theta + \dots \geq 0$$

At a contact  $\Psi(q, \theta) = 0$ .

Linear approximation:

$$\frac{\partial \Psi}{\partial q} \delta q + \frac{\partial \Psi}{\partial \theta} \delta \theta \geq 0$$

$$\boxed{W_n^T \delta q \geq J_n \delta \theta}$$

Quasistatic Manipulation of Frictionless Objects: Given  $\delta \theta, \dots$

$$\text{Min}_{\delta q} \quad c^T \delta q = -g_{ext}^T \delta q$$

$$\text{s.t. } W_n^T \delta q \geq \underbrace{J_n \delta \theta}_{\substack{\uparrow \\ A \quad x \quad \uparrow \\ \quad \quad \quad b}}$$

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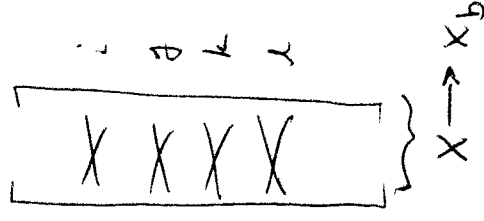
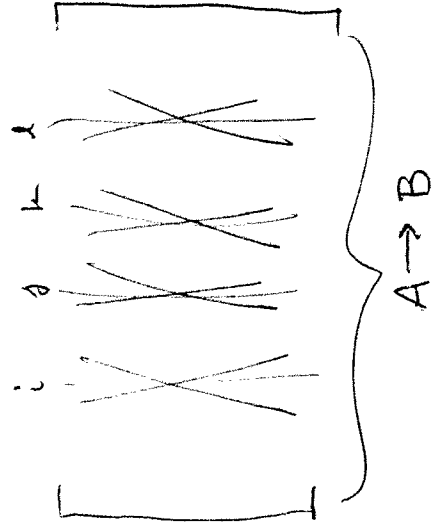
Note that  $s_q$  is a free variable.

We could use techniques from previous pages to cast in standard form.

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Definition

Basic Solution: Let  $B$  be an  $m \times m$  nonsingular submatrix formed by deleting columns of  $A$ . Then  $x_b = B^{-1}b$ ,  $x = (x_b, 0)$  is a basic solution.



$x_b$  are called basic variables.

Note  $A$  is  $m \times n$ . ~~It~~ It is assumed that  $n > m$

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It is assumed that  $A$  is full rank

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If  $A$  is not full row rank, then either rows of  $A$  can be eliminated or no solution exists.

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Definition: Feasible Solution: Any vector  $x$  satisfying

$Ax = b, x \geq 0$  is called a feasible solution.

Def: If  $x$  is basic & feasible, then it is called a basic feasible solution.

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simple  
Return to example on page (4.1).

$x^*$  is basic & feasible

$$B = [2]$$

$$x_b = x_2$$

$$x^* = \begin{bmatrix} 0 \\ 1.5 \end{bmatrix} \leftarrow x_b$$



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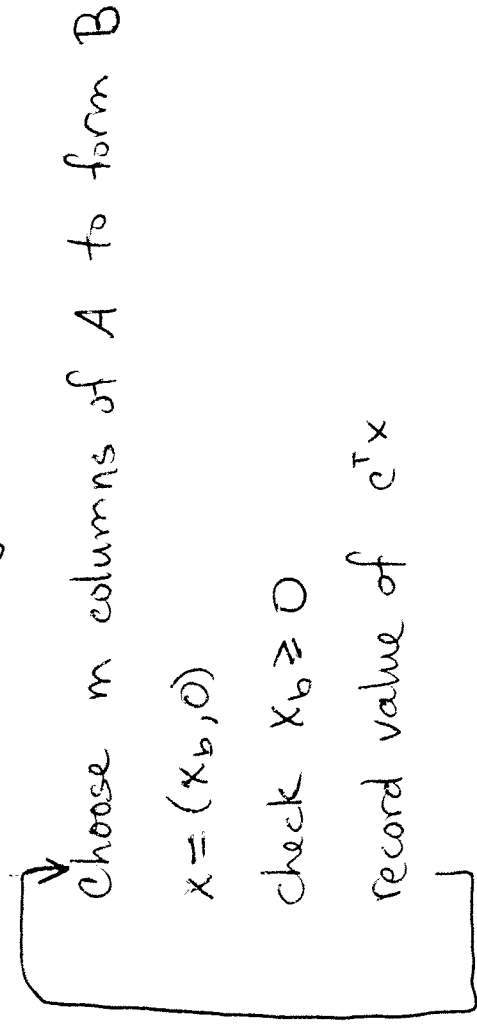
## Fundamental Thm of L.P.

Given an LP in standard form where

$A$  is  $(m \times n)$  and of rank  $m$

- i.) if there is a feasible solution, there is a basic feasible solution
  - ii.) if there is an optimal feasible sdn, there is an optimal basic feasible solution.
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## Enumerative Solution Alg. for LP.



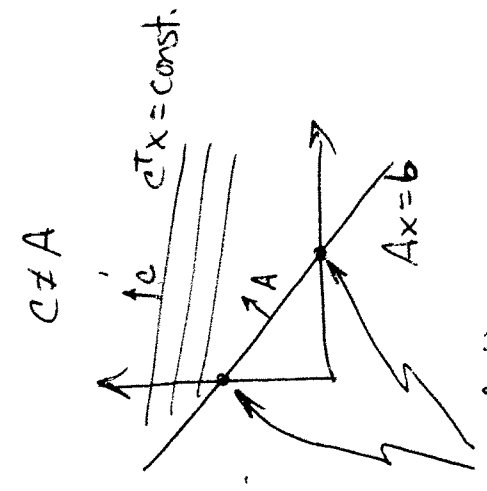
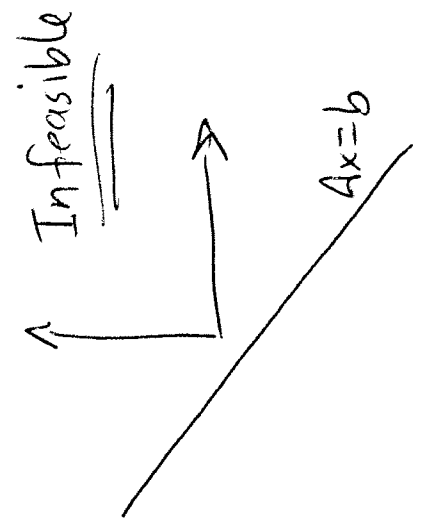
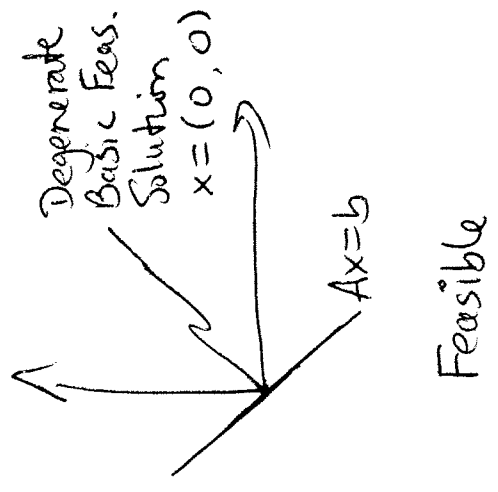
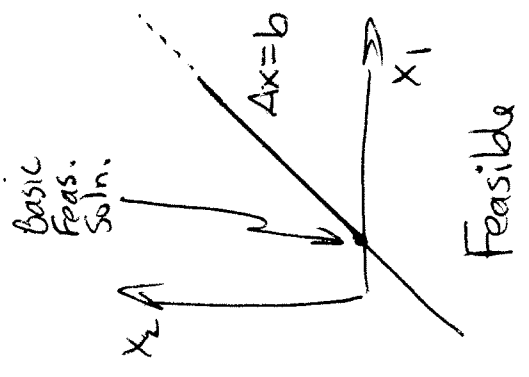
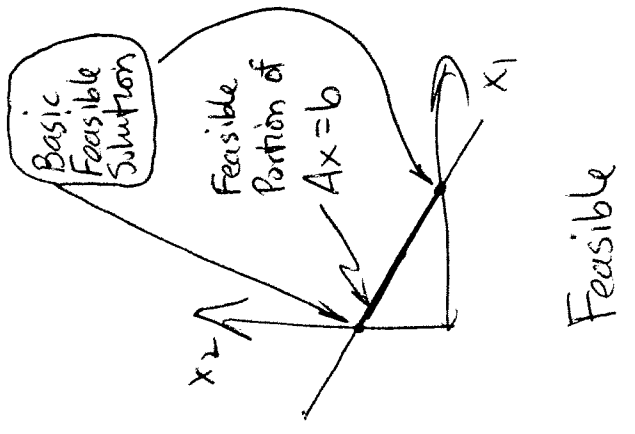
There are  $\binom{n}{m}$  passes required.  $\frac{n!}{m!(n-m)!}$

Enumeration is exponential in  $n$ .

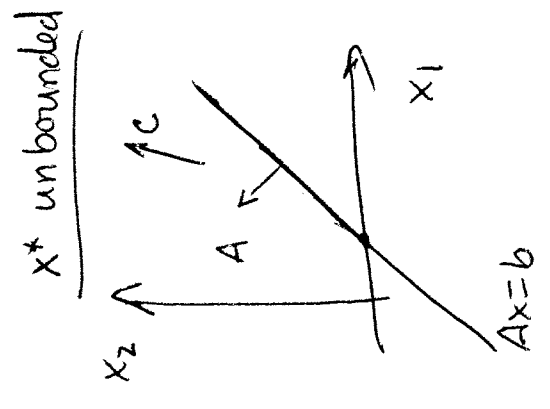
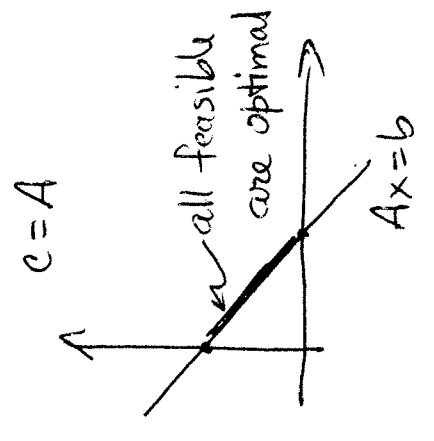
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one of these is  $x^*$



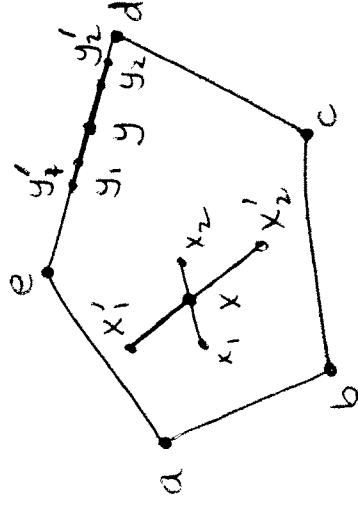
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## Convexity

Def: Extreme point: In a convex set  $C$ , a point  $x$  is an extreme point if there are no two distinct points  $x_1, x_2$  in  $C \ni x = \alpha x_1 + (1-\alpha)x_2$  for some  $\alpha, 0 < \alpha < 1$ .



Example:

$x$  is not extreme

$y$  is not extreme

$a, b, c, d, e$  are extreme pts.

Theorem: Let  $A$  be an  $m \times n$  matrix of rank  $m$  and  $b$  be an element of  $\mathbb{R}^m$ . Let  $K$  be a convex polytope consisting of all  $n$ -vectors  $x$  satisfying

$$\left. \begin{array}{l} Ax = b \\ x \geq 0 \end{array} \right\} K = \{x \mid Ax = b, x \geq 0\}$$

A vector  $x$  is an extreme point of  $K$  iff  $x$  is a basic feasible solution of  $Ax = b, x \geq 0$ .

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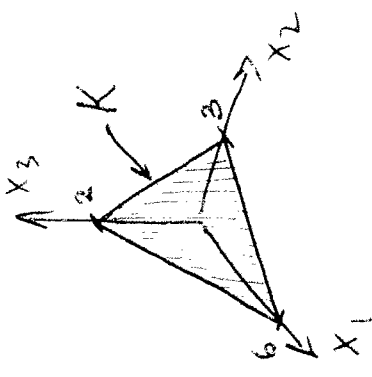
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Example

$$A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \quad b = 6$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\left. \begin{array}{l} Ax = b \\ x \geq 0 \end{array} \right\} = K =$$



How many basic solutions?

$$\binom{3}{1} = 3$$

$$B=1 \Rightarrow x = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$$

$$B=2 \Rightarrow x = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

$$B=3 \Rightarrow \dots -x = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

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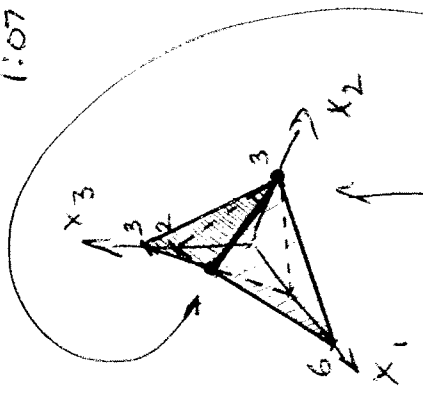
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$$\text{Let } A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

How many basic feasible solutions?

$$\binom{3}{2} = 3$$



$$\underline{x_3=0} \quad \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

$$\underline{x_2=0} \quad \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 1.5 \\ 0 \\ 1.5 \end{bmatrix}$$

$$\underline{x_1=0} \quad \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

Duplicate  
Basic  
Solutions

Similar example:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \text{basic solutions}$$

$$\left\{ \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

~~Not Feasible~~

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## Solution Algs for LP's

### Dantzig Method

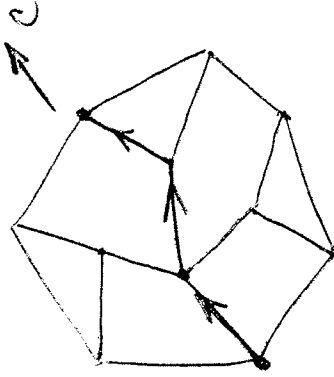
Simplex - search extreme points.

Worst-case performance is exponential in number of unknowns

Klee & Minty constructed an example with ~~2~~ a polytope with  $2^n$  vertices such that the Simplex Alg. visited all vertices before finding a solution. 1972.

Average-case performance is proportional to the number of ~~vertices~~ constraints  $2m - 3m$  steps

Graphical view of Simplex Algorithm



Move from vertex to vertex trying to increase  $c^T x$

Start

Simplex moves are also known as pivots.

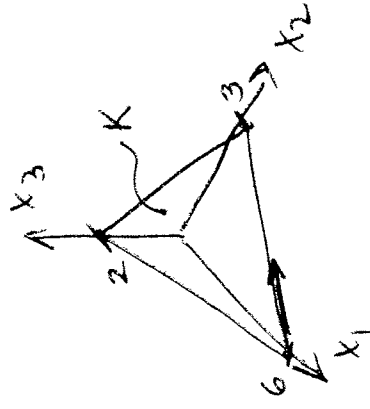
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Consider simple problem

$$\text{Min } \begin{bmatrix} 0 & -\frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\text{s.t. } x_1 + 2x_2 + 3x_3 = 6$$



Start with  $x_1$  basic,  $x_2, x_3$  nonbasic

i.e. Solve

$$x_1 + 2x_2 + 3x_3 = 6$$

$$x_2 = 0$$

$$x_3 = 0$$

$$\begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$$

$$e^T x = 0$$

Which variable should be basic next?

Equivalently, along which edge of  $K$  should we move?

Assume  $x_2$  becomes basic &  $x_1$  becomes nonbasic.

The edge is in the null space of

$$x_1 + 2x_2 + 3x_3 = 6$$

$$x_3 = 0$$

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parametric eq. of edge in  $x_1, x_2$ , plane.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1/2 \\ 0 \end{bmatrix} \alpha = \begin{bmatrix} 6-\alpha \\ 1/2\alpha \\ 0 \end{bmatrix} \geq 0 \Rightarrow 0 \leq \alpha \leq 6$$

Suppose  $x_3$  becomes basic

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1/3 \end{bmatrix} \beta = \begin{bmatrix} 6-\beta \\ 0 \\ 1/3\beta \end{bmatrix} \geq 0 \Rightarrow 0 \leq \beta \leq 6$$

Which pivot is better?

Consider ~~decrease~~ increase of  $e^T x$ .

$$\Delta(e^T x) = \begin{bmatrix} 0 & -1/2 & -1 \end{bmatrix} \begin{bmatrix} 6-\alpha \\ 1/2\alpha \\ 0 \end{bmatrix} = -1/4 \alpha \xrightarrow{\text{Biggest Change}} -1.5$$

$$\Delta(e^T x) = \begin{bmatrix} 0 & -1/2 & -1 \end{bmatrix} \begin{bmatrix} 6-\beta \\ 0 \\ 1/3\beta \end{bmatrix} = -1/3\beta \Rightarrow \boxed{-2.0}$$



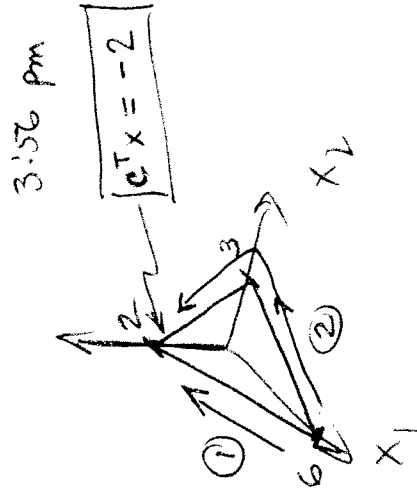
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Two paths to optimum

Path ① is direct

Path ② takes two steps



How do we know when we've arrived at the global minimum?

Look out 1-D null spaces from current point.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3/2 \\ -1 \end{bmatrix} \alpha, \quad 0 \leq \alpha \leq 2$$

$$\Delta(c^T x) = \begin{bmatrix} 0 & -1/2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 3/2 \\ -1 \end{bmatrix} \alpha = \frac{1}{4} \alpha$$

Note: Any movement of  $x$  along

the edge (i.e.,  $\alpha > 0$ ) increases  $c^T x$ .

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \beta, \quad 0 \leq \beta \leq 2 \Rightarrow \Delta(c^T x) = \beta$$

$\beta > 0$  increases  $c^T x$ .

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## Primal / Dual LPs

The soln of an LP generally includes  $x$  &  $\lambda$  where  $\lambda$  is the vector of Lagrange multipliers.

Values of Lagrange multipliers indicate how rapidly and in which direction the object function changes ~~is~~ due to constraint violation.

### PRIMAL

$$\begin{aligned} \text{Min } c^T x \\ \text{s.t. } Ax = b \\ x \geq 0 \end{aligned}$$

### DUAL

$$\begin{aligned} \text{Max } b^T \lambda \\ \text{s.t. } A^T \lambda \leq c \end{aligned}$$

Return to Manipulation Problem.

with some algebraic manipulation ...

(work done by contact forces)

$$\text{Min}_{\lambda_n} \sum_0^T J_n^T \lambda_n$$

$$\text{s.t. } W_n \lambda_n = -J_{\text{ext}}$$

(equilibrium conditions w/ unilateral contacts)

$$\text{Max}_{s_q} J_{\text{ext}}^T s_q$$

(Work done by gravity)

$$\text{s.t. } W_n^T s_q \geq J_n s_0$$

(Nonpenetration)