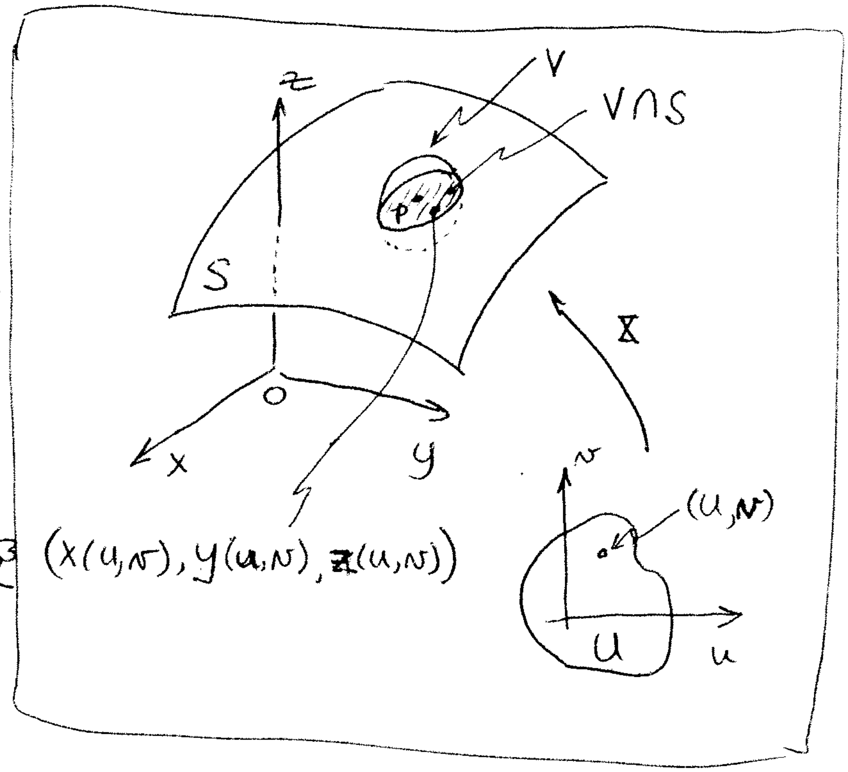
Do Carmo Chapter 2

2-2 Regular Surfaces; ~~and~~ Inverse Images of Regular Values

Crudely speaking, a regular surface is formed by taking planar patches, deforming them, and knitting them together such that there are no corners or edges.

<u>DEF 1</u>: A subset $S \subset \mathbb{R}^3$ is a regular surface if, for each $p \in S$, $\exists$ a neighborhood $V \subset \mathbb{R}^3$ and a map $\mathbf{x}: U \to V \cap S$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap S \subset \mathbb{R}^3$ such that:



1. $\mathbf{x}$ is differentiable.

   i.e. $\mathbf{x}(u,v) = (x(u,v), y(u,v), z(u,z))$ $\quad (u,v) \in U$

   The functions x, y, z have continuous partial derivatives of all orders in U

2. $\mathbf{x}$ is a homeomorphism.

   Since $\mathbf{x}$ is continuous, $\mathbf{x}^{-1}: V \cap S \to U$ which is continuous

3/31/06
②
9:25 AM

3. $\mathbf{X}$ must satisfy the regularity condition

i.e. $\forall\, q \in U$, the differential $dx_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one.

The mapping $\mathbf{X}$ is called a parametrization or a system of local coordinates in a neighborhood of p.

The nhbd $V \cap S$ of p is call a coordinate neighborhood.

Further explanation of third condition

4/1/06
9:46 Am

Choose cannonical bases

Parameter space basis $\{e_1, e_2\}$

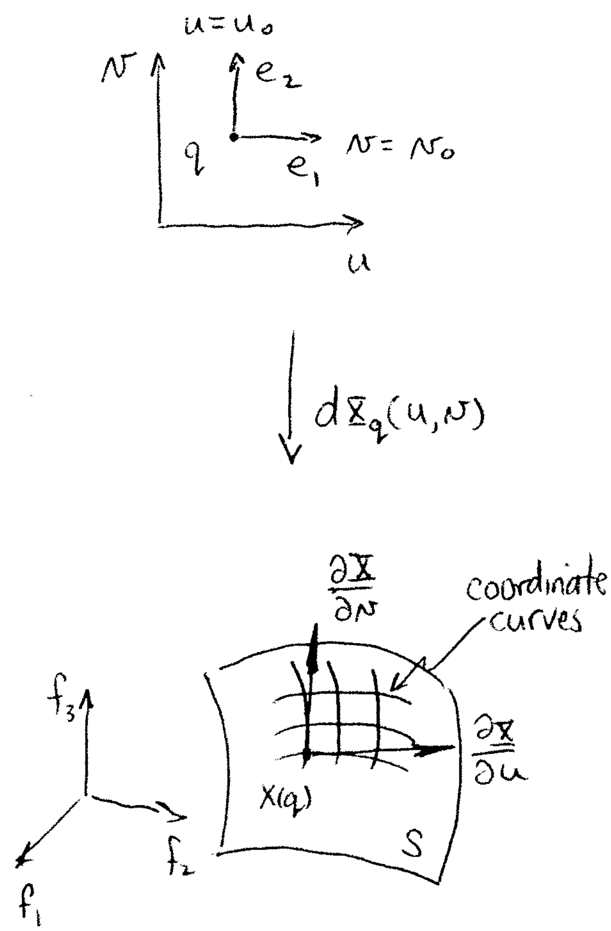
$$e_1 = (1, 0) \qquad e_2 = (0, 1)$$

$\mathbb{R}^3$ basis $\{f_1, f_2, f_3\}$

$$f_1 = (1,0,0) \quad f_2 = (0,1,0)$$
$$f_3 = (0,0,1)$$

The differential map $d\mathbf{X}_q$ is a linear map at q

$$d\mathbf{X}_q = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} = \left(d\mathbf{X}_q(e_1),\ d\mathbf{X}_q(e_2) \right)$$



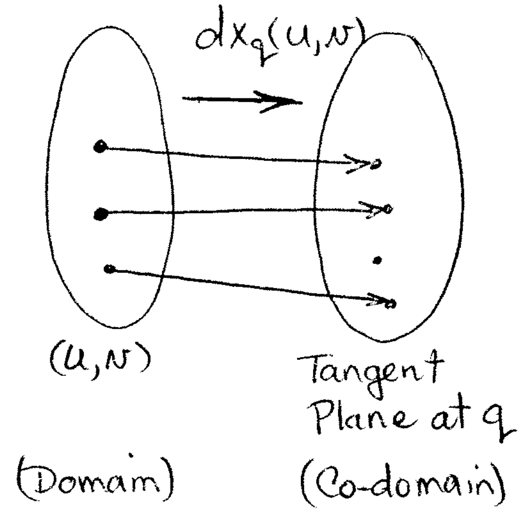
<u>DEF</u> Coordinate curve: a curve on S defined by $\underline{X}(u,v)$ holding u or v constant while the other varies.

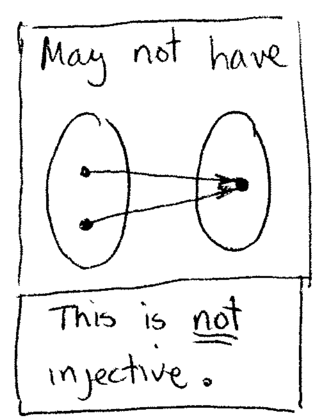
Returning to condition 3, the differential map must be 1-1.

i.e. for every point in the co-domain, there is at most one point in the domain.

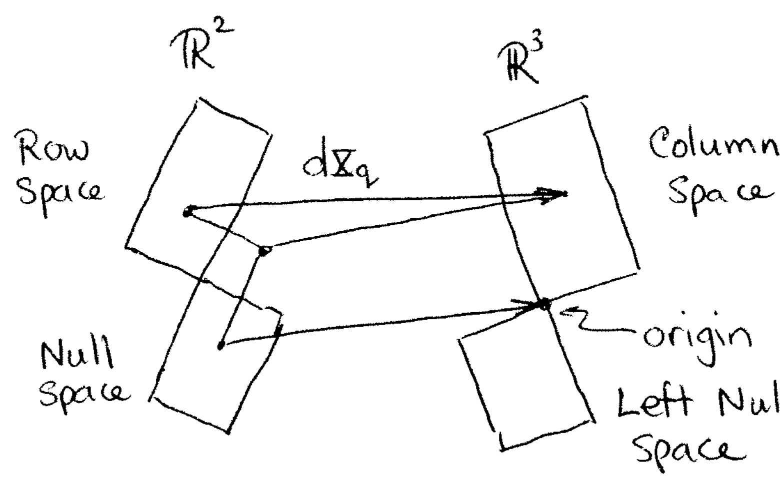
i.e. distinct argument yield distinct results

i.e. the map is <u>injective</u>





$d\underline{X}_q$ is a linear map of dimension (3×2).



4/1/06

10:16 AM

$d\mathbf{X}_q$ is 1-1 iff NullSpace($d\mathbf{X}_q$) is zero dimensional

In other words the rank of $d\mathbf{X}_q$ must be **2**.

$\Rightarrow$ At least 1 2x2 determinant of $d\mathbf{X}_q$ must be nonzero

$\Rightarrow$ $\dfrac{\partial \mathbf{X}}{\partial u} \wedge \dfrac{\partial \mathbf{X}}{\partial v} \neq 0$

$\Rightarrow$ $\dfrac{\partial \mathbf{X}}{\partial u}, \dfrac{\partial \mathbf{X}}{\partial v}$ are linearly independent

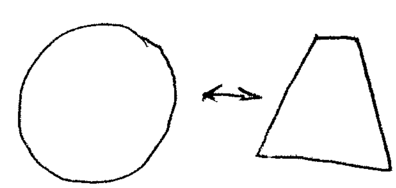
$\Rightarrow$ span of columns of $d\mathbf{X}_q$ is a plane

Condition 2 - $\mathbf{X}$ is a homeomorphism (or topological isomorphism)

homeos = identical

morphe = shape

A geometric object is homeomorphic to another geometric object if one can be made from the other by continuous stretching and bending.

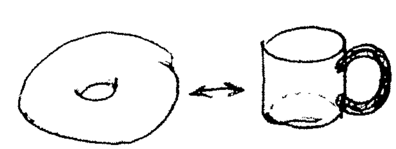


A fcn f between two topological spaces X & Y is a homeomorphism if it satisfies :

f is 1-1 and onto (bijection)

f is continuous

f^{-1} is continuous



$(-1,1) \longleftrightarrow (-\infty,\infty)$

Insights:

Condition 1 - we need differentiability to study S from perspective of differential geometry.

Condition 2 - essential to proving that certain objects properties defined by parameterization do NOT depend on the specific parameterization, but on S only.

Condition 3 - guarantees existence of a tangent plane at every point on S.

Example: Let $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\} \Leftarrow$ unit sphere

Is the unit sphere a regular surface?

$\mathbf{X}_1 : U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$\mathbf{X}_1(x,y) = (x, y, \sqrt{1-(x^2+y^2)}) \qquad (x,y) \in U$$

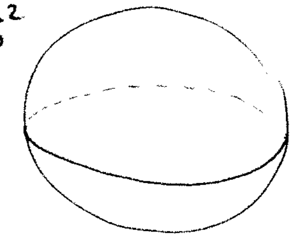
where $\mathbb{R}^2 = \{(x,y,z) \in \mathbb{R}^3 \mid z = 0\}$

$U = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\}$

verify that $\mathbf{X}_1$ is a parametrization of S^2

Note that $\mathbf{X}_1(U)$ is the upper open hemisphere.

S^2



4/1/06

6

10:49 AM

Condition 1

Clearly x, y, $\sqrt{1-(x^2+y^2)}$ are all differentiable to all orders on U

Condition 3

$$d\mathbf{X}_1 = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$$

From def of U, we see u plays roll of x

v " " " y

$$\therefore \frac{\partial x}{\partial u} = \frac{\partial x}{\partial x} = 1$$

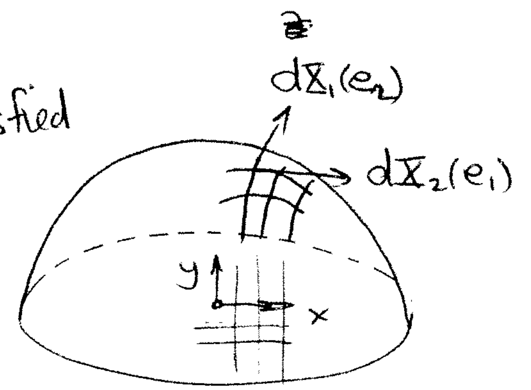
$$\frac{\partial x}{\partial y} = \frac{\partial y}{\partial x} = 0$$

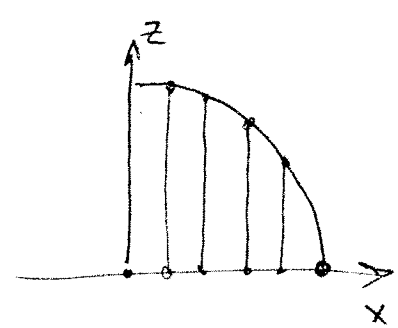
$$\frac{\partial y}{\partial y} = 1$$

$$d\mathbf{X}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ (\cdot) & (\cdot) \end{pmatrix} \Longleftarrow$$

$\therefore$ $d\mathbf{X}_1$ is 1-1 & Cond. 3 is satisfied

Note that $d\mathbf{X}_1(e_1)$ and $d\mathbf{X}_1(e_2)$ are not necessarily orthogonal





Condition 2

Clearly the map $\mathbf{X}_1$ is 1-1 and onto.

Is $\mathbf{X}_1$ continuous? Yes

Is $\mathbf{X}_1^{-1}$ continuous? Yes

We can cover the entire sphere with similar parametrizations, but to cover whole sphere we need 6 overlapping parametrizations.

$$\mathbf{X}_2(x,y) = \left(x, y, -\sqrt{1-(x^2+y^2)}\right)$$

$$\mathbf{X}_3(x,z) = \left(x, \sqrt{1-(x^2+z^2)}, z\right)$$

$$\vdots$$

$$\mathbf{X}_6(y,z) = \left(-\sqrt{1-(y^2+z^2)}, y, z\right)$$

Determining if a given subset of $\mathbb{R}^3$ is a regular surface directly from the definition is laborious. The following 4 propositions simplify this task.

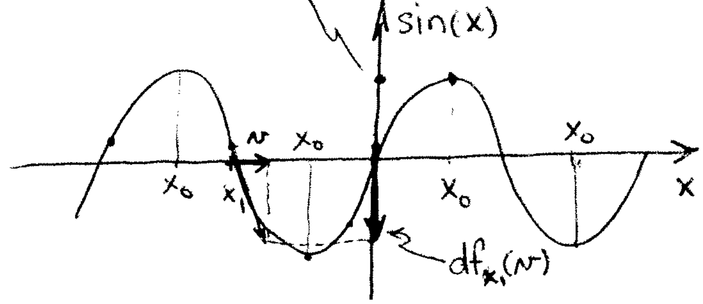
<u>Proposition 1</u> If $f: U \to \mathbb{R}$ is a differentiable function in an open set U of $\mathbb{R}^2$, then the graph of f, $(x, y, f(x,y))$, for $(x,y) \in U$ is a regular surface.

<u>DEF 2</u>: Given a differentiable map $F: U \subset \mathbb{R}^n \to \mathbb{R}^m$ defined in an open set U of $\mathbb{R}^n$, p is a critical point of F if $dF_p: \mathbb{R}^n \to \mathbb{R}^m$ is <u>not</u> surjective (onto). The image $F(p) \in \mathbb{R}^m$ of a critical point is called a <u>critical value of F</u>. All other points in $\mathbb{R}^m$ are <u>regular values</u> of F.

Example: $f(x) = \sin(x) : U \in \mathbb{R} \longrightarrow \mathbb{R}$

$df_{x_0}(v) = 0$

$f(x_0) = 1$



$U = (-\infty, \infty)$

$f'(x) = \cos(x)$

$f'(x_0) = \cos(x_0) = 0 \Rightarrow x_0$ is a critical point of the map.

The critical points of $\sin(x)$ are : $x_0 = (2p-1)\frac{\pi}{2} \quad p \in \mathbb{Z}$

The critical values of the map are : ± 1

The regular values of the map are : $(-\infty,-1), (-1,1), (1,\infty)$

$df_{x_0}(v)$ carries all vectors $v \in \mathbb{R}$ to the zero vector.

Note that the differential map from $\mathbb{R}^1$ to $\mathbb{R}^1$ is simply a scaling.

If $f : U \subset \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a differentiable function, then df_p applied to the vector $(1,0,0)$ is the tangent at $f(p)$ to the curve $x \longrightarrow f(x, y_0, z_0)$

It follows that df_p in the basis $(1\,0\,0), (0\,1\,0), (0\,0\,1)$ is

$$df_p = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

4/1/06

9

5:03 PM

In this case, df_p is not surjective.

Equivalently $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ are not simultaneously 0.

$$\boxed{\begin{array}{l}\text{Punch line: } a \in f(U) \text{ is a regular value of} \\ f: U \subset \mathbb{R}^3 \to \mathbb{R} \ \underline{\text{iff}}\ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \text{ do not} \\ \text{vanish simultaneously at any point in the inverse} \\ \text{image } f^{-1}(a).\end{array}}$$

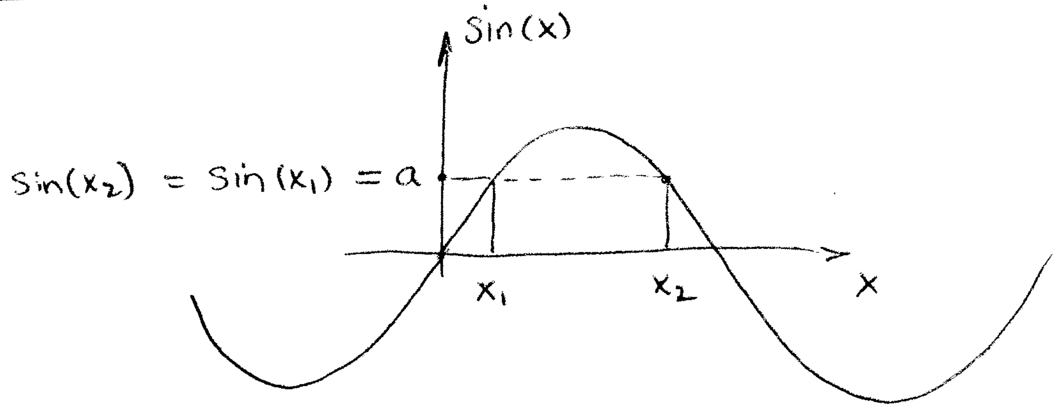


Image of $x_1 = \sin(x_1) = a$

Inverse image of $a = \{ x_1 + p2\pi, x_2 + p2\pi \}, p \in \mathbb{Z}$

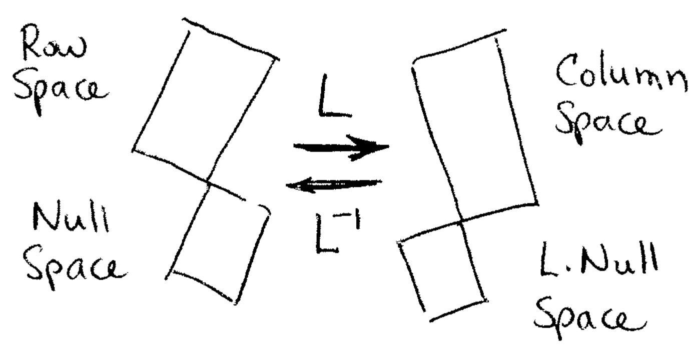
a is a regular value since $\frac{\partial f}{\partial x} \neq 0 \ \forall x = f^{-1}(a)$.

Linear maps

L is 1-1 if Null Sp. = 0
(injective)

L is onto if L.Null Sp. = 0
(surjective)

L is 1-1 & onto if N.Sp & L.N.Sp = 0
(bijective)



4/1/06
5:24 pm
10

Proposition 2.

If $f: U \subset \mathbb{R}^3 \to \mathbb{R}$ is a differentiable fcn and $a \in f(U)$ is a regular value of f, then $f^{-1}(a)$ is a regular surface in $\mathbb{R}^3$.

Example: Ellipsoid : $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is a reg. surface.

This ellipsoid is the inverse image of 0 where f is given as :

$$f(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 .$$

That is, the ellipsoid is defined as $f^{-1}(0)$.

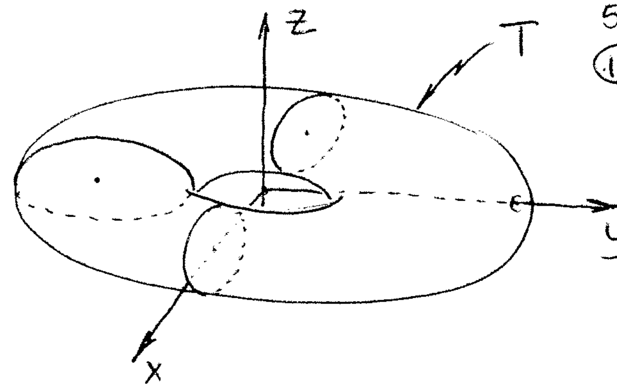
Assuming $a, b, c, \neq 0$, we have

$$df = \begin{pmatrix} \frac{2x}{a^2} \\ \frac{2y}{b^2} \\ \frac{2z}{c^2} \end{pmatrix} . \quad \text{This equals} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ iff } x = y = z = 0 .$$

Since $(0,0,0)$ is not on the ellipsoid, all points ~~on~~ on the ellipsoid are regular and hence the ellipsoid is a regular surface.

Example: Torus, T

Generate by rotating a circle of radius r about the z-axis.



Let S^1 be circle in yz-plane with center $(0, a, 0)$ $a > r$.

S^1 is given by $(y-a)^2 + z^2 = r^2$

T is given by $z^2 = r^2 - (\sqrt{x^2+y^2} - a)^2$

$\therefore$ T is the inverse image of r^2 by the function

$$f(x,y,z) = z^2 + (\sqrt{x^2+y^2} - a)^2$$

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= \frac{2x(\sqrt{x^2+y^2} - a)}{\sqrt{x^2+y^2}} \\ \frac{\partial f}{\partial y} &= \frac{2y(\ \cdot\ )}{\sqrt{\quad}} \\ \frac{\partial f}{\partial z} &= 2z \end{aligned}\right\} \Rightarrow$$

f is differentiable except for the z-axis $(x=y=0)$.
$\therefore$ the z-axis is the set of critical points of f.

Since r^2 is a regular value of f, T is a regular surface.

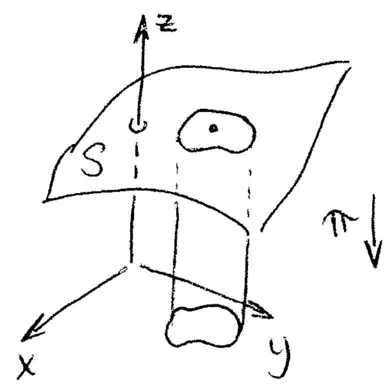
Recall that r^2 is a regular value of f, because $f^{-1}(r^2) = T$ and $T \cap$ z-axis $= \emptyset$

4/1/06
12
7:36 pm

<u>Proposition 3</u>: Let $S \subset \mathbb{R}^3$ be a regular surface and $p \in S$. Then $\exists$ a nbhd V of p in $S \ni V$ is the graph of a differentiable fcn which has one of the following three forms:

$z = f(x,y)$, $y = g(x,z)$, $x = h(y,z)$.

Roughly speaking, prop. 3 says that any patch ~~on S~~ (small enough) on S can be parametrized by two of the 3 cannonical basis coordinates.



<u>Proposition 4</u>: Let $p \in S$ be a point on a regular surface S and let $\mathbf{X}: U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a map with $p \in \mathbf{X}(U) \ni$ conditions 1 & 3 of Definition 1 hold. Assume that $\mathbf{X}$ is one-to-one. Then $\mathbf{X}^{-1}$ is continuous.

Example: One-sheeted Cone, $\mathcal{C}$,

$$z = +\sqrt{x^2+y^2} \quad , \ (x,y) \in \mathbb{R}^2$$

is not a regular surface.

Why?
The natural parameterization, $(x,y) \xrightarrow{z} (x, y, +\sqrt{x^2+y^2})$

is not differentiable at $(x,y) = (0,0)$ since

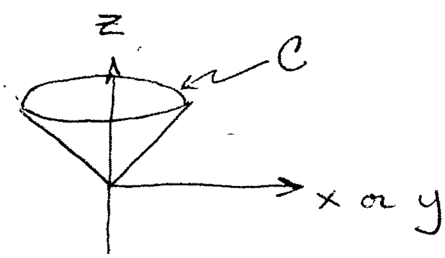
$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}\Big|_{(0,0)} = \text{undefined}$$

$$\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}\Big|_{(0,0)} = \text{undefined}$$

However this fact is not sufficient to conclude that the cone is not a regular surface, since there may be some other parametrization that is differentiable.

Use Prop 3. Assume $\mathcal{C}$ is a regular surface. Then
At $(0,0,0) \in \mathcal{C}$, $\mathcal{C}$ must be the graph of a differentiable function, $z = f(x,y)$, $x = g(y,z)$, or $y = h(x,z)$.

consider $\left.\begin{array}{r} x = g(y,z) \\ y = h(x,z) \end{array}\right\}$ Not 1-1, so g & h cannot be functions.



4/1/06

(14)

8:17 pm

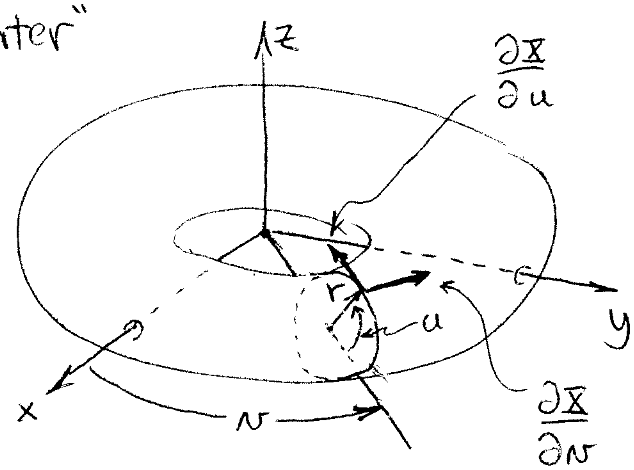
The remaining form, $z = f(x,y)$, would have to agree with $z = \sqrt{x^2+y^2}$ in the neighborhood of $(0,0,0)$. Since z is not differentiable, no function $z = f(x,y)$ exist!

<u>Example</u> A parametrization of the torus T is:

$$\mathbf{X}(u,v) = ((r\cos(u) + a)\cos(v), (r\cos(u) + a)\sin(v), r\sin(u))$$

$$U = \{0 < u < 2\pi, \ 0 < v < 2\pi\}$$

a = radius of "center" of torus



<u>DEF 1; COND 1</u>

$x(u,v)$, $y(u,v)$, $z(u,v)$ must be differentiable to all orders. This is obvious

<u>DEF 1; COND 3</u>

This can be checked algebraically, but I will use a geometric argument. We must have that $\frac{\partial \mathbf{X}}{\partial u} \neq 0$, $\frac{\partial \mathbf{X}}{\partial v} \neq 0$, and $\frac{\partial \mathbf{X}}{\partial u} \wedge \frac{\partial \mathbf{X}}{\partial v} \neq 0$. Equivalently, $\frac{\partial \mathbf{X}}{\partial u}, \frac{\partial \mathbf{X}}{\partial v}$ may never be parallel or have zero magnitude. See figure.

4/1/06

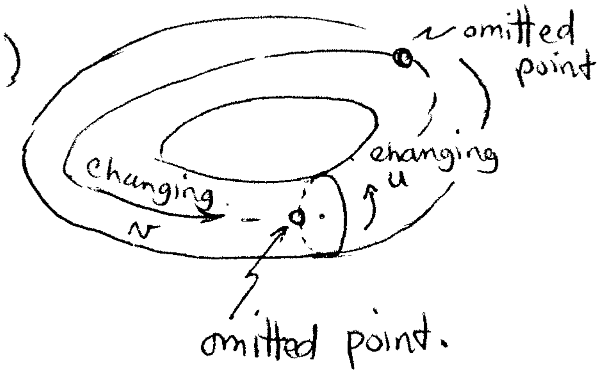
15

8:36 pm

Finally we must show that $\mathbf{X}$ is one-to-one.

This can be done algebraically, but is also clear from geometric arguments.

$$\mathbf{X}(u,v) = (\cdot \ , \ \cdot \ , \ \cdot)$$

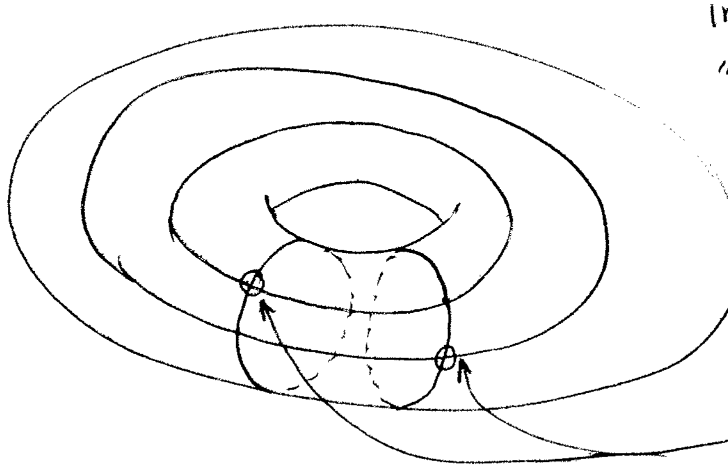


$$U \begin{cases} 0 < u < 2\pi \longrightarrow \mathbf{X}(u,v_0) \\ 0 < v < 2\pi \longrightarrow \mathbf{X}(u_0,v_0) \end{cases}$$

$$\mathbf{X}(U) = T - S' - S'$$

Claim T can be covered by 3 parametrization similar to that above.

Shift v & u intervals so that "uncovered" circles do not coincide.



Two uncovered points remain.

One more shifted parametrization will cover those points.