Practice / Test Problems

1. Given the parameterized curve

$$\alpha(s) = \left(a \cos\left(\frac{s}{c}\right),\ a \sin\left(\frac{s}{c}\right),\ b\frac{s}{c}\right)$$

where $s, a, b, c \in \mathbb{R}$ and $c^2 = a^2 + b^2$

and a, b, c are constant

a. Show that the parameter s is arc length.

b. Determine the curvature and torsion of $\alpha(s)$.

c. Determine the osculating plane of $\alpha(s)$.

d. Show that lines containing $n(s)$ and passing thru $\alpha(s)$ meet the z-axis under a constant angle $\pi/2$.

e. Show that the tangent lines to $\alpha(s)$ make a constant angle with the z-axis

2. Show that the torsion of $\alpha(s)$ is given by

$$\tau(s) = -\frac{\alpha'(s) \wedge \alpha''(s) \cdot \alpha'''(s)}{k^2(s)}$$

3. Assume that all normals of a parameterized curve $\alpha(s)$ pass thru a fixed point. Prove that the trace of $\alpha(s)$ is contained in a circle.

4. Given the plane curve $\alpha(\gamma) : I \rightarrow \mathbb{R}^2$

$$\alpha(\gamma) = \left(a \cos^3(\gamma),\ a \sin^3(\gamma)\right) \quad , \quad a \in \mathbb{R}$$

(Note γ is <u>not</u> arc length.)

a is constant

$a > 0$

a. Determine $t(\gamma)$

b. Determine $n(\gamma)$

c. Determine where $n(\gamma)$ & $t(\gamma)$ are not well defined

d. Determine the length of the trace of $\alpha(\gamma)$ for $\gamma \in [0, \pi/2]$. (This part is too complicated for a test question. Lots of trig. & algebra & calculus.)

Homework Problems

1. Consider the map

$$\alpha(s) = \begin{cases} (s,\ 0,\ e^{-1/s^2}) & , \ s > 0 \\ (s,\ e^{-1/s^2},\ 0) & , \ s < 0 \\ (0,\ 0,\ 0) & , \ s = 0 \end{cases}$$

a. Prove that $\alpha(s)$ is a differentiable curve.

b. Prove that $\alpha(s)$ is regular for all s and that the curvature $k(s) \neq$, for $s \neq 0$, $s \neq \pm\sqrt{2/3}$, and $k(0) = 0$.

c. Show that the limit of the osculating planes as $s \to 0$, $s > 0$ is the plane $y = 0$, but that the limit as $s \to 0$, $s < 0$ is $z = 0$.

d. Show that $\tau(s)$ can be defined so that $\tau \equiv 0$, even though $\alpha(s)$ is not a plane curve.