Universality of noise-induced resilience restoration in spatially-extended ecological systems

Cheng Ma, Gyorgy Korniss, Boleslaw K. Szymanski, Jianxi Gao

Oct. 20, 2022

Dynamics, topology, and resilience I

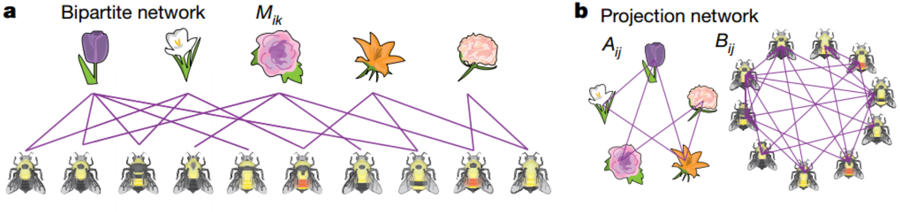


Figure 1: Illustration of mutualistic relationship between plants and pollinators. (a) The mutualistic interaction M_{ij} between bees and flowers. (b) From M_{ij}, one can construct two mutualistic networks by linking pairs of plants that share mutual pollinators (A_{ij}), or pollinators that share mutual plants (B_{ij}).

Dynamics, topology, and resilience II

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = F(x_i) + \sum_{j=1}^{N} A_{ij} G(x_i, x_j) \tag{1}$$

$$\begin{aligned} F(x_i) &= B_i + x_i \left(1 - \frac{x_i}{K_i}\right)\left(\frac{x_i}{C_i} - 1\right) \\ G(x_i, x_j) &= \frac{x_i x_j}{D_i + E_i x_i + H_j x_j} \end{aligned} \tag{2}$$

1. x_i: the abundance of species i (the node state) .
2. B_i: the incoming migration rate of i from neighboring ecosystems.
3. K_i: carrying capacity.
4. C_i: the Allee effect, accounting for the negative growth rate with low abundance.

5. D_i, E_i, H_i: the parameters of the response function which represents mutualistic relationship, indicating that $j's$ positive contribution to x_i is bounded for large x_i or x_j.

The parameters for numerical simulations are set as $B_i = 0.1$, $C_i = 1$, $K_i = 5$, $D_i = 5$, $E_i = 0.9$, $H_j = 0.1$ for all species.

Dynamics, topology, and resilience IV

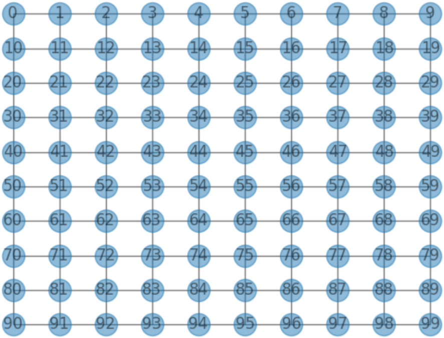


Figure 2: Interaction topology: 10×10 2D lattice. The number of nodes $N = 100$.

One variable mapping (Gao *et al.* [1]):

$$\frac{\mathrm{d}x_{eff}}{\mathrm{d}t} = B + x_{eff}\left(1 - \frac{x_{eff}}{K}\right)\left(\frac{x_{eff}}{C} - 1\right) + \beta_{eff}\frac{x_{eff}^2}{D + Ex_{eff} + Hx_{eff}} \tag{3}$$

where x_{eff} is the average state of the entire system, and β_{eff} is the average interaction strength.

$$x_{eff} = \frac{\mathbf{1}^T A\boldsymbol{x}}{\mathbf{1}^T A\mathbf{1}} \tag{4}$$

$$\beta_{eff} = \frac{\mathbf{1}^T A\boldsymbol{k}^{in}}{\mathbf{1}^T A\mathbf{1}} \tag{5}$$

Dynamics, topology, and resilience VI

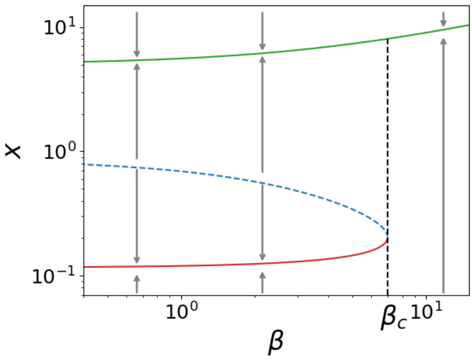


Figure 3: The resilience diagram for one variable system. There is a critical threshold of phase transition $\beta_c \approx 7$. $\beta < \beta_c$, there are two stable states x_L, x_H and one unstable state x_u. $\beta > \beta_c$, there is only one stable state x_H.

Motivation I

Question: For the case when all species/nodes are attracted to the low state x_L, with the weak interaction strength ($\beta < \beta_c$), can we recover the system to the high state x_H?

Solutions:

1. Increase interaction strength β beyond β_c, and then the system naturally evolves to the desired state x_H.
2. In the real system, fluctuations are ubiquitous. We use independent Gaussian noise $\eta_i(t)$ to simulate random fluctuations of species abundance.

$$\langle \eta_i(t)\eta_j(t') \rangle = \sigma^2 \delta_{ij}\delta(t - t') \tag{6}$$

After adding noise, the dynamics becomes

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = F(x_i) + \sum_{j=1}^{N} A_{ij} G(x_i, x_j) + \eta_i \tag{7}$$

Motivation II

Our research motivation is to study whether random fluctuations can drive the system back to desired state, and how long it takes for such transition.

Noise-induced nucleation and transition I

To generalize the analysis, we define the normalized state ρ_i between 0 and 1.

$$\rho_i(t) = \frac{x_i(t) - x_L}{x_H - x_L} \tag{8}$$

The state of the entire system can be described by the average of $\rho_i(t)$.

$$\rho(t) = \langle \rho_i(t) \rangle_N = \frac{1}{N} \sum_{i=1}^{N} \rho_i(t) \tag{9}$$

In the presence of noise, the low state is $\rho \approx 0$ and the high state is $\rho \approx 1$. Initially, $\rho(t = 0) = 0$.

Noise-induced nucleation and transition II

Simulation setup: All the species are in the low state ρ_L initially.

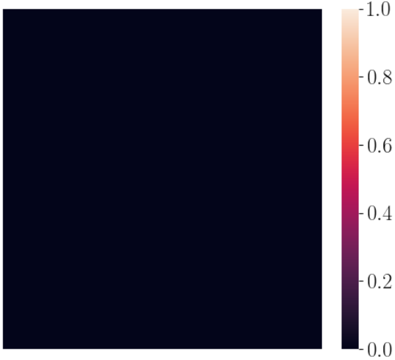
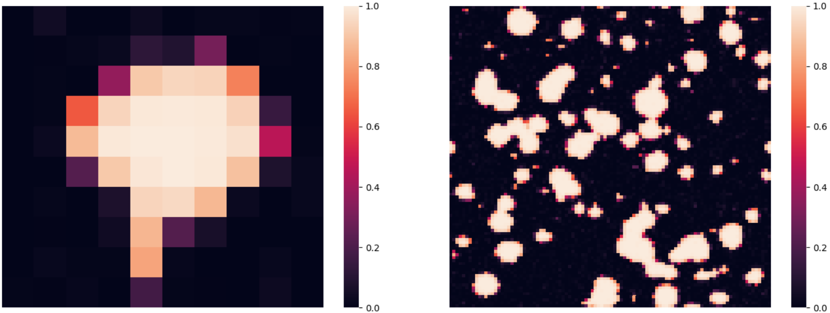


Figure 4: $\beta = 4$, $N = 100$, $\sigma = 0.08$.

Noise-induced nucleation and transition III

Figure 5: $\beta = 4$, $N = 10000$, $\sigma = 0.08$.

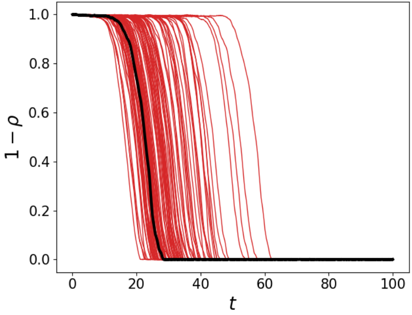
Take-away message: Random noise can drive the system from the extinct state x_L to the active state x_H, leading to resilience recovery.



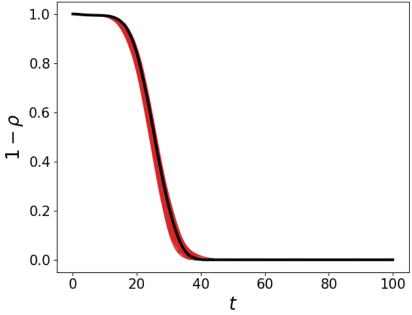
(a) $N = 100$ (b) $N = 10000$

Figure 6: **There are two cluster modes: single-cluster mode and multi-cluster mode. Noise strength $\sigma = 0.1$.** (a) $N = 100$, single-cluster mode. (b) $N = 10000$, multi-cluster mode.

Cluster mode and transition time II



(a) $N = 100$



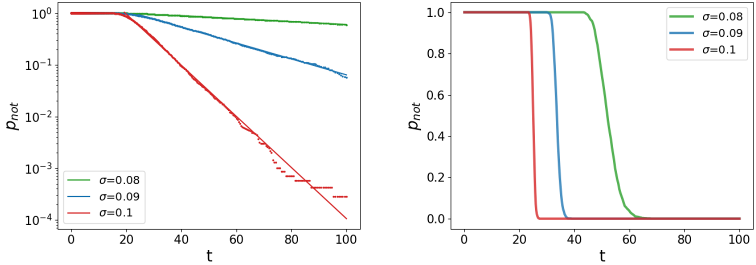
(b) $N = 10000$

Figure 7: **The evolution of the global state ρ for 100 realizations. Noise strength $\sigma = 0.1$.** (a) $N = 100$, single-cluster mode. (b) $N = 10000$, multi-cluster mode.

Cluster mode and transition time III

1. The first question has been answered: random fluctuations can drive the system to the desired state. We are also interested in the time required to complete the recovery process.
2. To quantify the time for the system to switch to the high state, the transition time τ is defined as the time when ρ just exceeds $\frac{1}{2}$, which is also the half lifetime of the initial state.
3. Because of random perturbations, it is inherently random when the first cluster appears. One would expect lifetime τ differs for different realizations.

Cluster mode and transition time IV



(a) $N = 100$ (b) $N = 10000$

Figure 8: **The probability distribution of waiting time P_{not}, defined as the fraction of random realizations that have not been recovered by time t.** (a) $N = 100$ single-cluster mode. (b) $N = 10000$ multi-cluster mode.

Take-away message: The larger system generates more clusters, thus spatial self-averaging reduces the randomness of transition time τ.

The effects of system size and noise strength on the average transition time $\langle \tau \rangle$ I

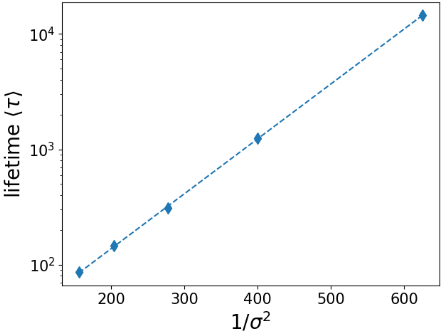


Figure 9: $\langle \tau \rangle$ is averaged over 1000 realizations.

For the single variable system, $\langle \tau \rangle \sim e^{\frac{c}{\sigma^2}}$.

The effects of system size and noise strength on the average transition time $\langle \tau \rangle$ II

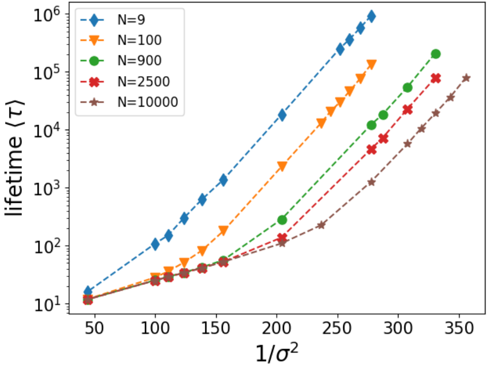


Figure 10: $\langle \tau \rangle$ is averaged over 1000 realizations for different system sizes N and noise strengths σ.

Crossover and finite-size scaling I

According to Avrami's homogeneous nucleation theory [2],

$$\langle \tau \rangle \sim \begin{cases} \frac{e^{\frac{c}{\sigma^2}}}{N}, & N^{\frac{1}{2}} \ll R_0 \quad \text{(single-cluster mode)} \\ e^{\frac{c}{3\sigma^2}}, & N^{\frac{1}{2}} \gg R_0 \quad \text{(multi-cluster mode)} \end{cases}, \tag{10}$$

where $R_0 \sim e^{\frac{c}{3\sigma^2}}$ is the typical distance between separate clusters (and $N^{1/2}$ is the linear size of the two-dimensional lattice).

Crossover and finite-size scaling II

By constructing a scaling function with the following asymptotic behavior,

$$f(x) \sim \begin{cases} x^2, & x \gg 1 \\ \text{const.}, & x \ll 1 \end{cases}, \tag{11}$$

where $x = R_0/N^{\frac{1}{2}}$, one can capture the average lifetime of *any* system size and noise values (including the crossover between the single-cluster and multi-cluster regimes),

$$\langle \tau \rangle = e^{\frac{c}{3\sigma^2}} f(R_0/N^{\frac{1}{2}}) = e^{\frac{c}{3\sigma^2}} f(e^{\frac{c}{3\sigma^2}}/N^{\frac{1}{2}})\ . \tag{12}$$

Plot $\langle \tau \rangle e^{-\frac{c}{3\sigma^2}}$ vs. $e^{\frac{c}{3\sigma^2}}/N^{\frac{1}{2}}$.

Crossover and finite-size scaling III

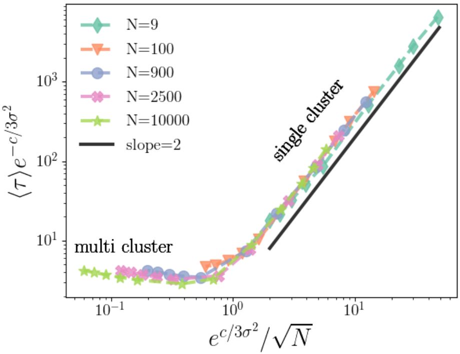


Figure 11: Finite-size scaling of two cluster modes.

Conclusions

1. Random fluctuations can recover the system from the undesired state to the desired state, leading to the resilience recovery.
2. Two cluster modes (single-cluster and multi-cluster modes) are decided by system size N and noise value σ, and they exhibit different transition patterns and lifetime features.
3. For the multi-cluster mode, the spatial-averaging effects reduce randomness, resulting in the deterministic evolutions.
4. The average transition time $\langle \tau \rangle$ for two cluster modes can be represented by a universal scaling law.

References

Jianxi Gao, Baruch Barzel, and Albert-Lszl Barabsi.
Universal resilience patterns in complex networks.
Nature, 530(7590):307–312, February 2016.
00411.

Gyorgy Korniss and Thomas Caraco.
Spatial dynamics of invasion: the geometry of introduced species.
Journal of Theoretical Biology, 233, 2005.

Appendix

For the single-cluster mode, the individual transition time τ varies a lot. The distribution of waiting time P_{not} is derived as

$$P_{not}(t) = \begin{cases} 1, & t \leq t_g \\ e^{-(t-t_g)/\langle t_n \rangle}, & t > t_g \end{cases}, \tag{13}$$

1. t_g represents the time needed for the global state ρ to exceed $\frac{1}{2}$ after the first cluster appears, which can be approximated as constant independent of system size and noise strength.
2. $\langle t_n \rangle$ is the average time elapsing until the first transition occurs from the initial states (i.e., the first cluster nucleates).