Frontiers of Network Science
Fall 2022

Class 17: Robustness II
(Chapter 8 in Textbook)

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based on slides by
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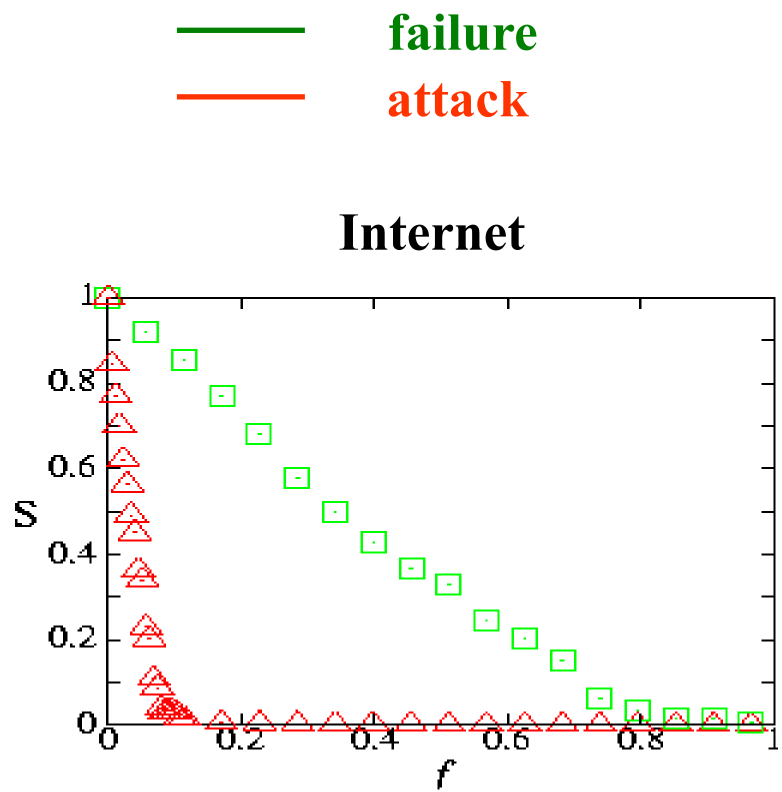
and Roberta Sinatra

www.BarabasiLab.com

SUMMARY: ACHILLES' HEEL OF COMPLEX NETWORKS

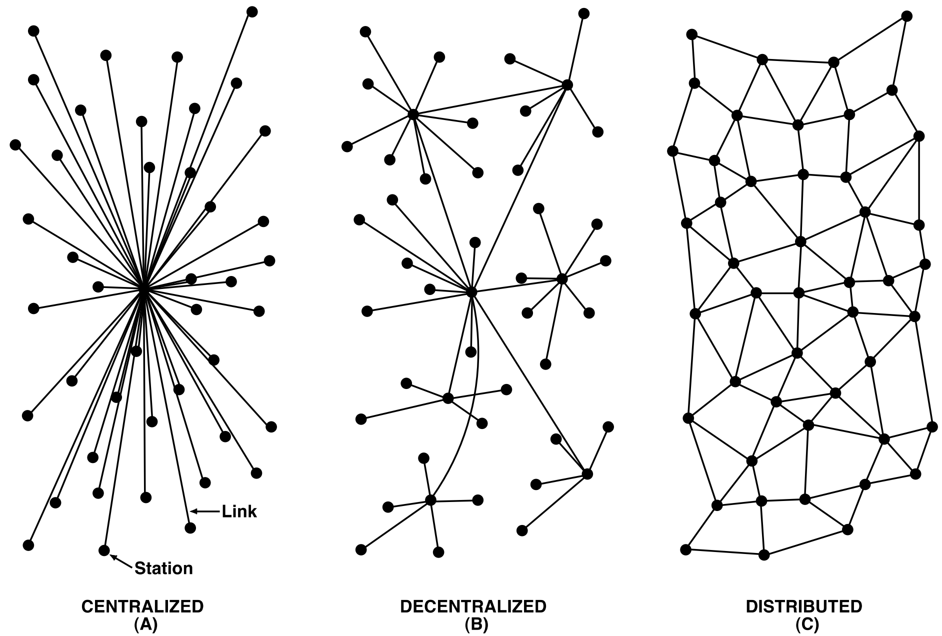


R. Albert, H. Jeong, A.L. Barabasi, *Nature* **406** 378 (2000)

HISTORICAL DETOUR: PAUL BARAN AND INTERNET

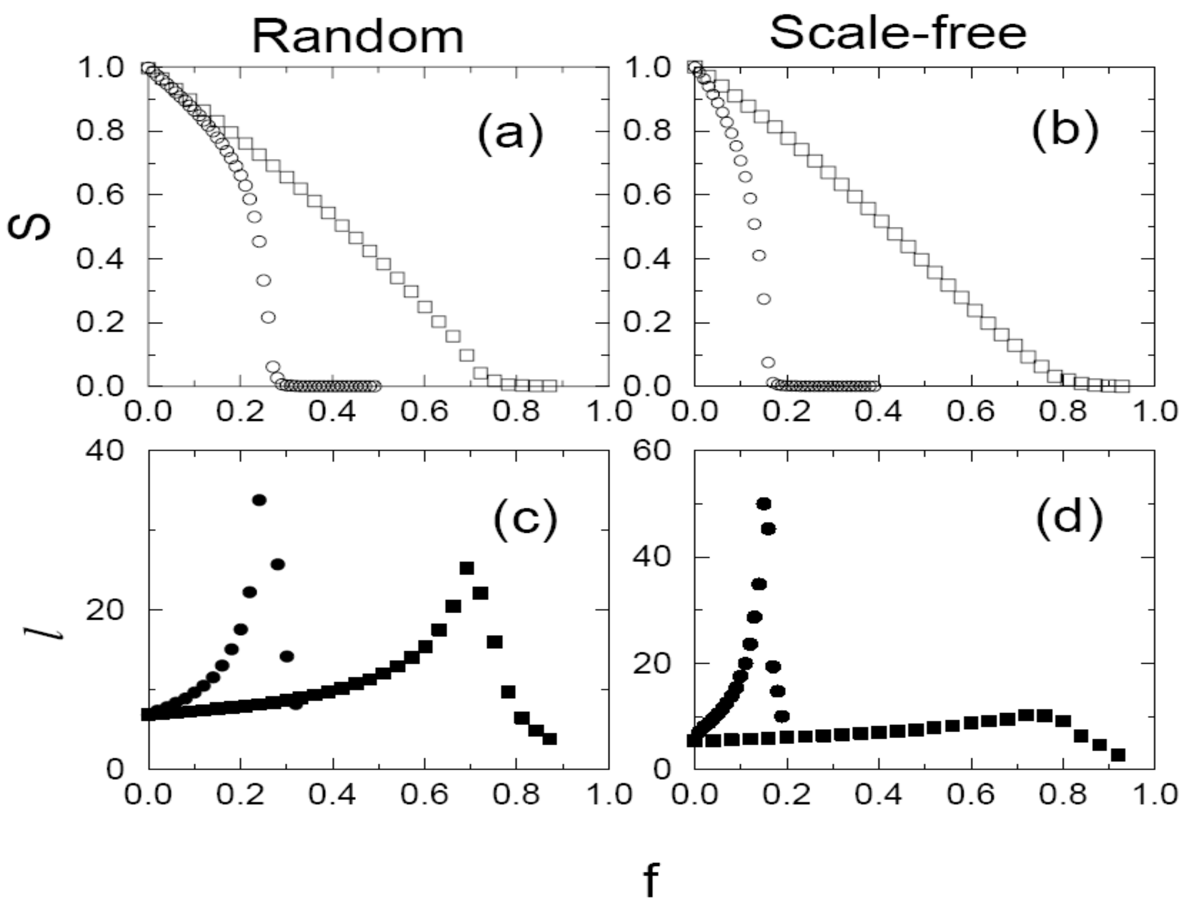
1958

A network of n-ary degree of connectivity that has n links per node was simulated



The simulation revealed that networks where n ≥ 3 had a significantly increased resilience against the large failure involving at least 50% node loss. Baran's insight gained from the simulation was that redundancy was the key.

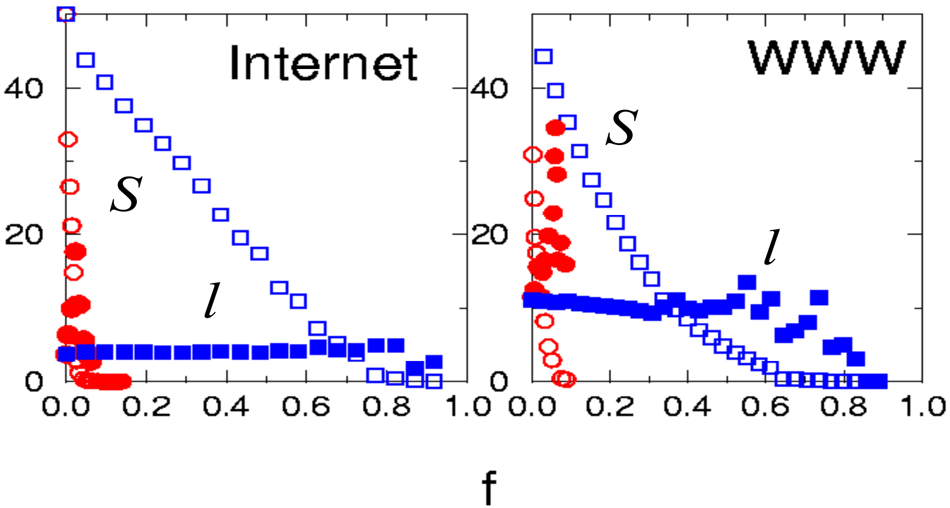
SCALE-FREE NETWORKS ARE MORE ERROR TOLERANT, BUT ALSO MORE VULNERABLE TO ATTACKS



- squares: random failure
- circles: targeted attack
- S surviving fraction of GC
- *l* average distance between nodes

Failures: little effect on the integrity of the network.
Attacks: fast breakdown

REAL SCALE-FREE NETWORKS SHOW THE SAME DUAL BEHAVIOR



- blue squares: random failure
- red circles: targeted attack
- open symbols: ***S*** *(size of surviving component)*
- filled symbols: ***l*** *(average distance)*

- break down if 5% of the nodes are eliminated selectively (always the highest degree node)
- resilient to the random failure of 50% of the nodes.

Similar results have been obtained for metabolic networks and food webs.

CASCADES

Potentially large events triggered by small initial shocks

- **Information cascades**

  social and economic systems

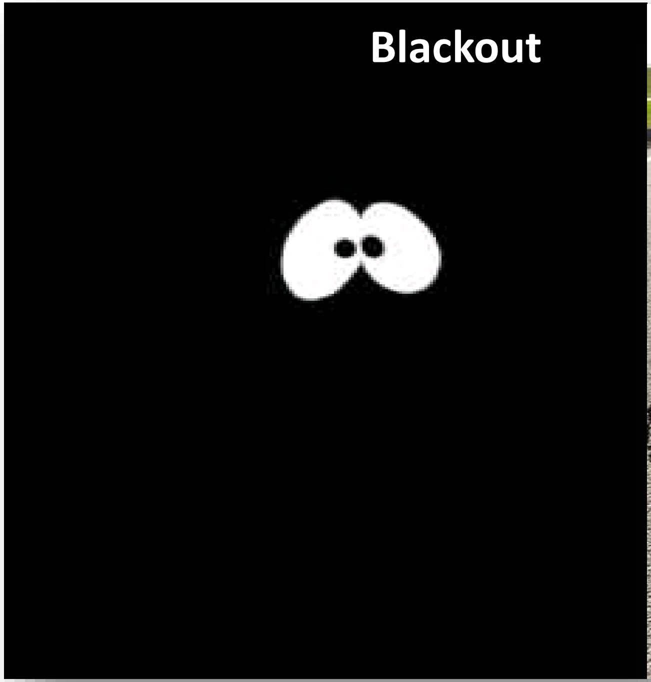
  diffusion of innovations

- **Cascading failures**

  infrastructural networks

  complex organizations

CASCADING FAILURES IN NATURE AND TECHNOLOGY



Flows of physical quantities

- congestions
- instabilities
- Overloads

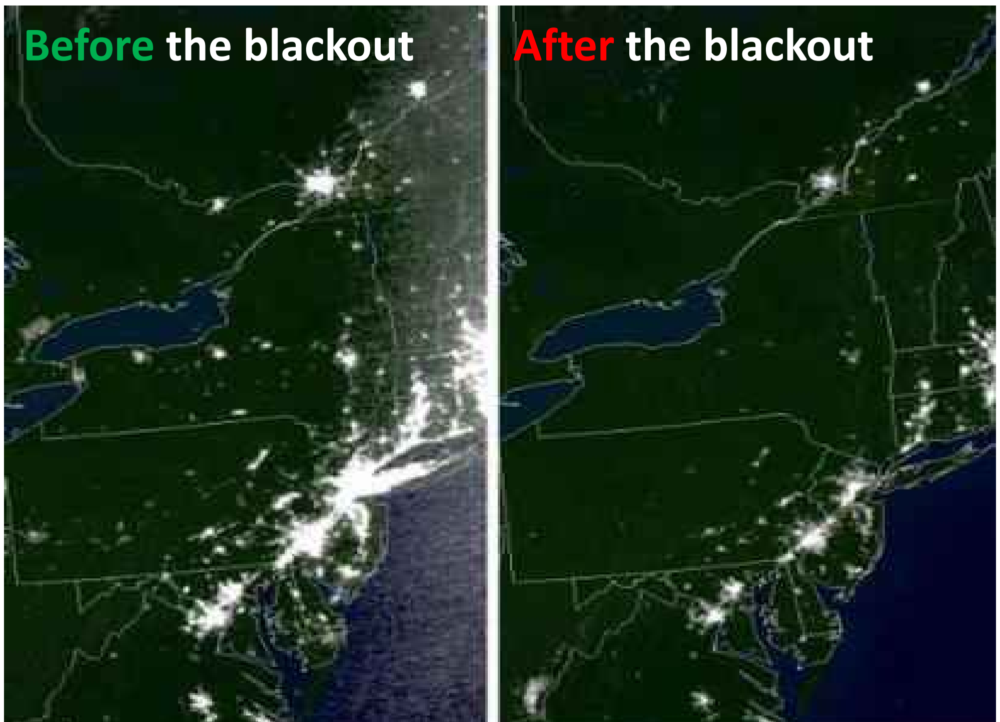
Cascades depend on

- Structure of the network
- Properties of the flow
- Properties of the net elements
- Breakdown mechanism

NORTHEAST BLACKOUT OF 2003

Origin

A 3,500 MW power surge (towards Ontario) affected the transmission grid at 4:10:39 p.m. EDT. (Aug-14-2003)



Consequences

More than 508 generating units at 265 power plants shut down during the outage. In the minutes before the event, the NYISO-managed power system was carrying 28,700 MW of load. At the height of the outage, the load had dropped to 5,716 MW, a loss of 80%.

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NUMBER 73

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$1.50 outside of
Metro Boston

The Boston Globe

A NEW WEAK

TODAY: *Partly sunny and colder. Hi 37-42. Low 27-32.*
TOMORROW: *Mostly sunny, milde High 42-47. Low 32-37.*
HIGH TIDE: *6:42 a.m., 7:25 p.m.*
SUNRISE: *6:59* SUNSET: *6:49*
FULL REPORT: PAGE B13

MONDAY, MARCH 14, 2011

Cascading disaster in Japan

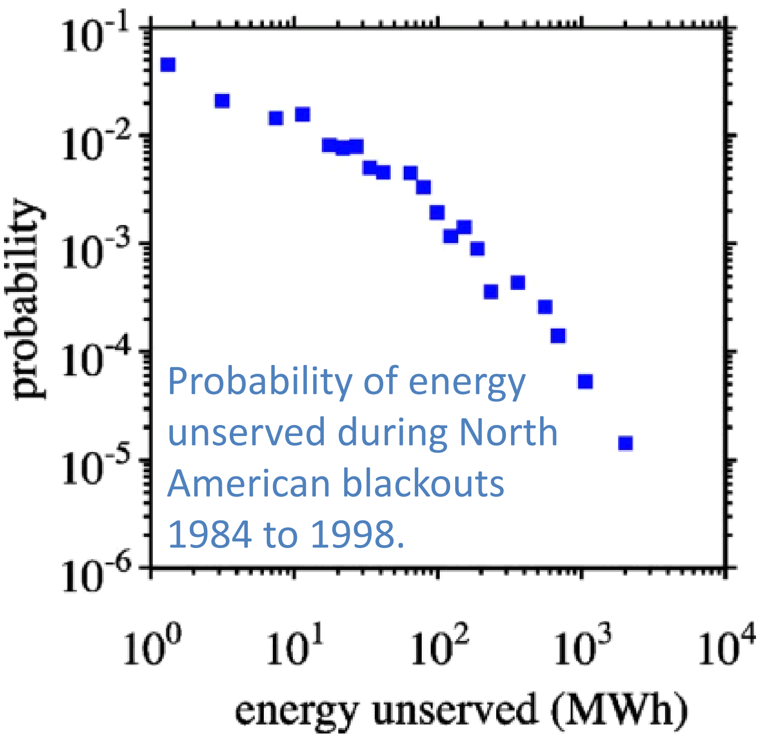
Blast shakes a second reactor death toll soar

By Martin Fackler and Mark McDonald
NEW YORK TIMES

SENDAI, Japan — Japan reel from a rapidly unfolding disaster epic scale yesterday, pummeled by t death toll, destruction, and homele ness caused by the earthquake a tsunami and new hazards from da aged nuclear reactors. The prime m ister called it Japan's worst crisis sir World War II.

Japan's $5 trillion economy, t world's third largest, was threaten with severe disruptions and partial p ralysis as many industries shut dov temporarily. The armed forces and v unteers mobilized for the far more gent crisis of finding survivors, evac ating residents near the strick power plants and caring for the v tims of the record 8.9 magnitu quake that struck on Friday.

CASCADES SIZE DISTRIBUTION OF BLACKOUTS



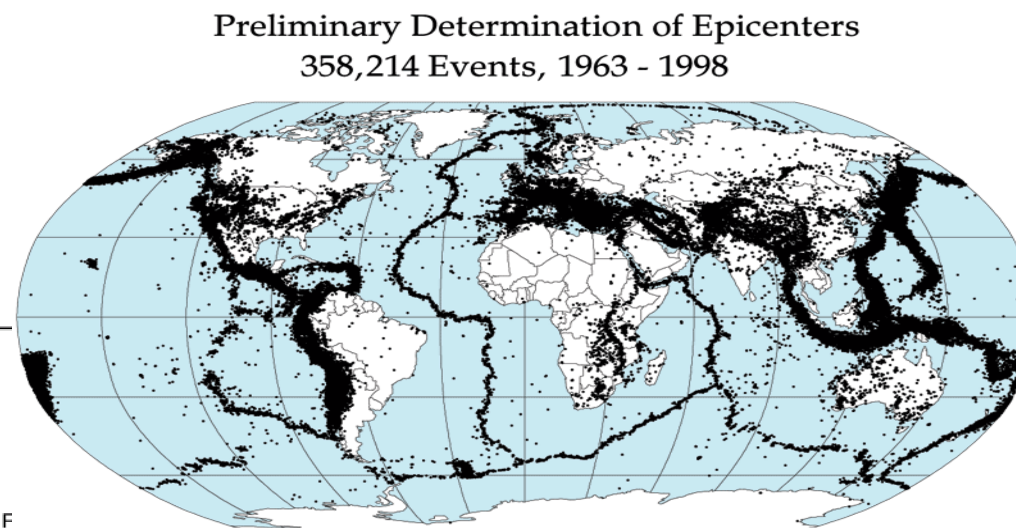
Unserved energy/power magnitude (*S*) distribution

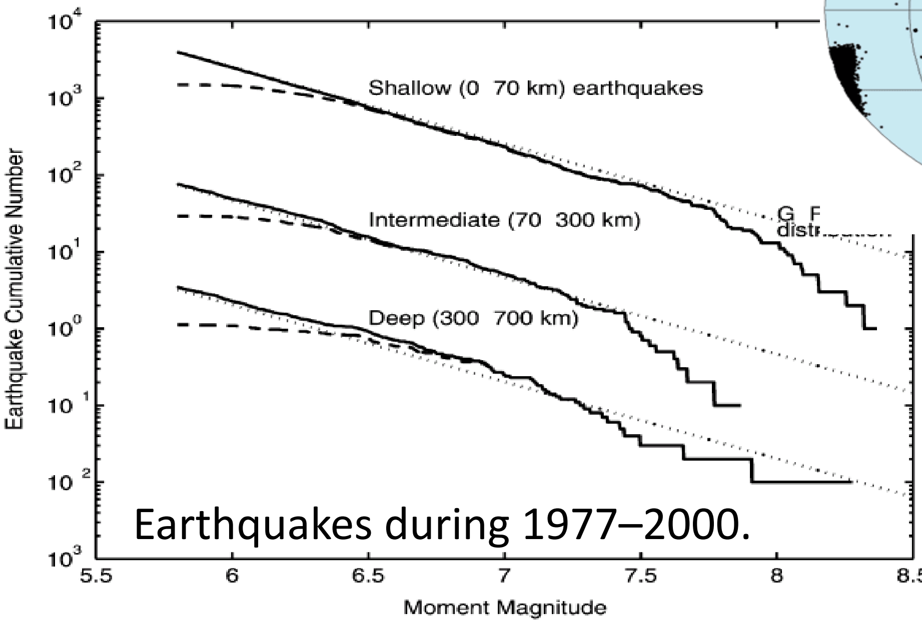
$$P(S) \sim S^{-\alpha},\ 1 < \alpha < 2$$

Source	Exponent	Quantity
North America	2.0	Power
Sweden	1.6	Energy
Norway	1.7	Power
New Zealand	1.6	Energy
China	1.8	Energy

I. Dobson, B. A. Carreras, V. E. Lynch, D. E. Newman, *CHAOS* 17, 026103 (2007)

CASCADES SIZE DISTRIBUTION OF EARTHQUAKES



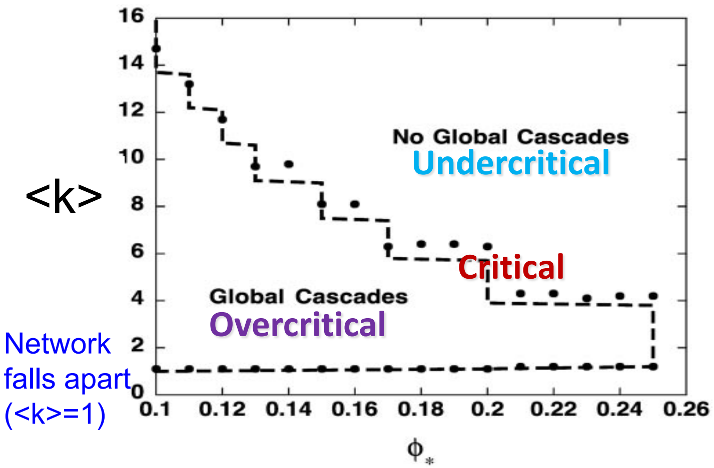


Earthquake size *S* distribution

$$P(S) \sim S^{-\alpha}, \alpha \approx 1.67$$

Y. Y. Kagan, *Phys. Earth Planet. Inter.* 135 (2–3), 173–209 (2003)

FAILURE PROPAGATION MODEL

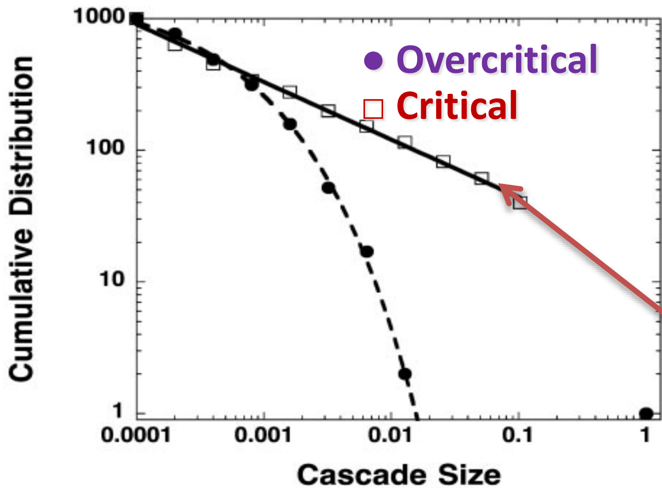


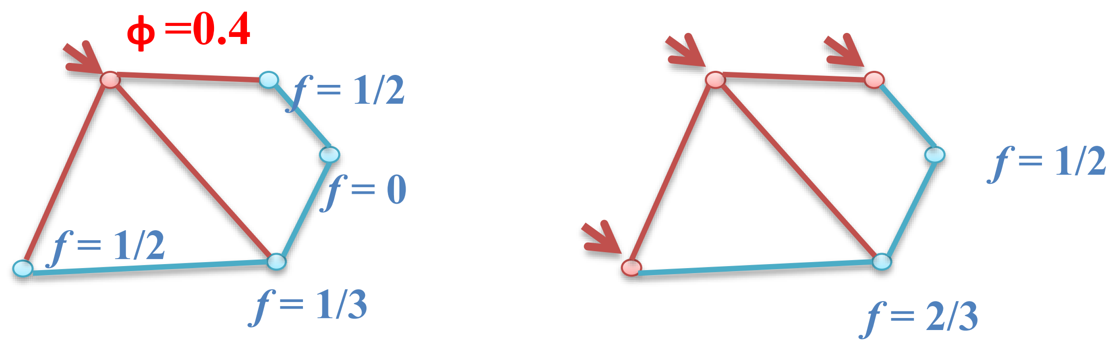
Initial Setup

- Random graph with *N* nodes
- Initially each node is functional.

Cascade

- Initiated by the failure of one node.
- f_i : fraction of failed neighbors of node *i*. Node *i* fails if f_i is greater than a global threshold ϕ.



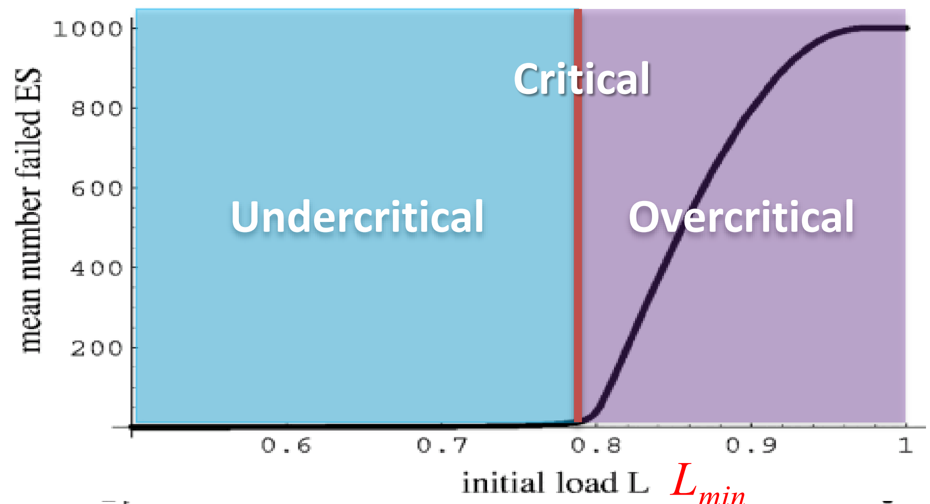


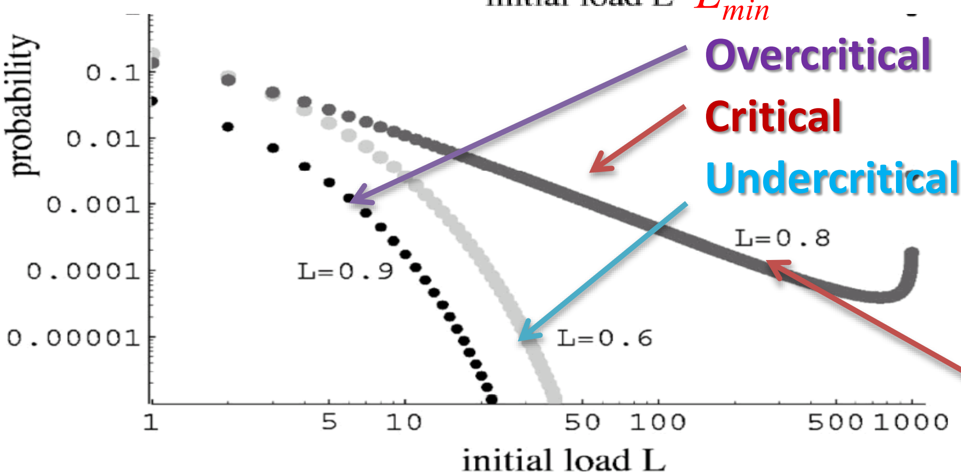
Erdos-Renyi network

$$P(S) \sim S^{-3/2}$$

D. Watts, *PNAS* 99, 5766-5771 (2002)

OVERLOAD MODEL



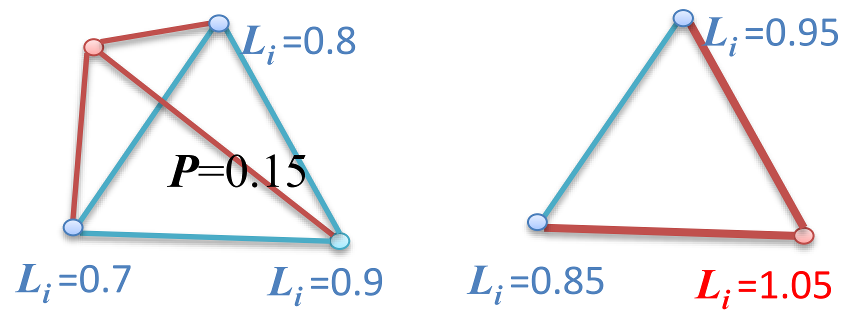


Initial Conditions

- *N* Components (complete graph)
- Each components has random initial load L_i drawn at random uniformly from $[L_{min}, 1]$.

Cascade

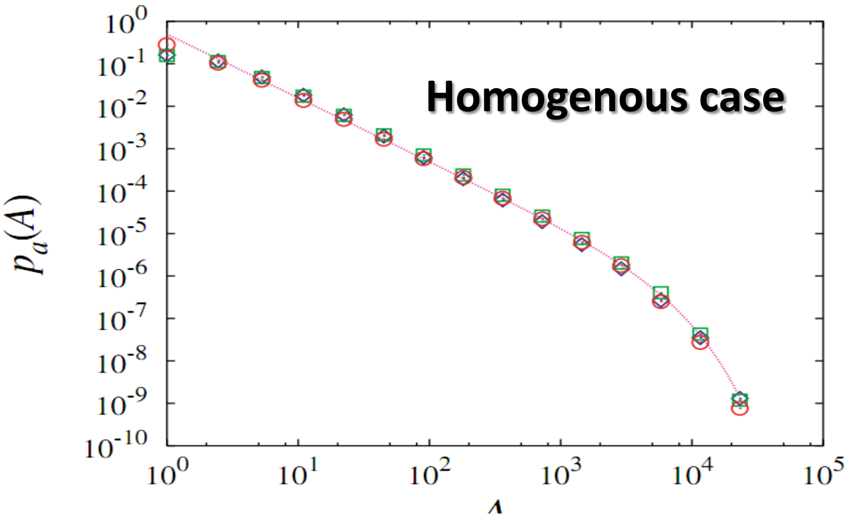
- Initiated by the failure of one component.
- Component fail when its load exceeds *1*.
- When a component fails, a fixed amount *P* is transferred to all the rests.

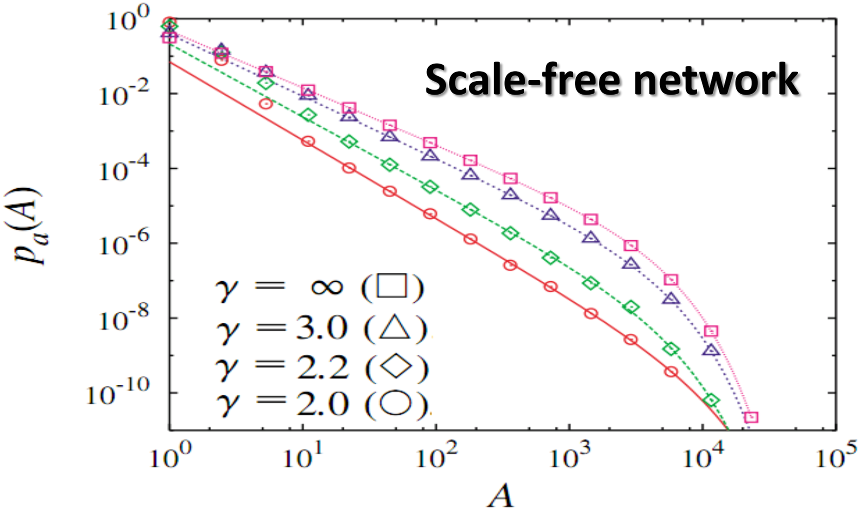


$$P(S) \sim S^{-3/2}$$

I. Dobson, B. A. Carreras, D. E. Newman, *Probab. Eng. Inform. Sci.* **19**, 15-32 (2005)

SELF-ORGANIZED CRITICALITY AKA SANDPILE MODEL





Initial Setup

- Random graph with N nodes
- Each node *i* has height $h_i = 0$.

Cascade

- At each time step, a grain is added at a randomly chosen node *i*: $h_i \leftarrow h_i + 1$
- If the height at the node *i* reaches a prescribed threshold $z_i = k_i$, then it becomes unstable and all the grains at the node topple to its adjacent nodes: $h_i = 0$ and $h_j \leftarrow h_j + 1$
- if *i* and *j* are connected.

Homogenous network: $\langle k^2 \rangle$ converges

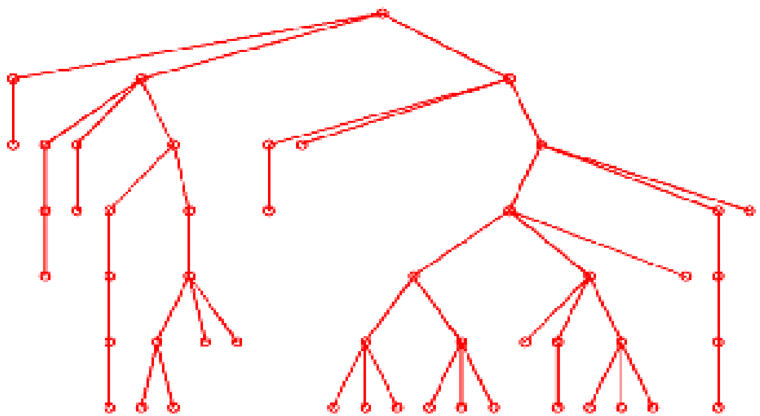
$$P(S) \sim S^{-3/2}$$

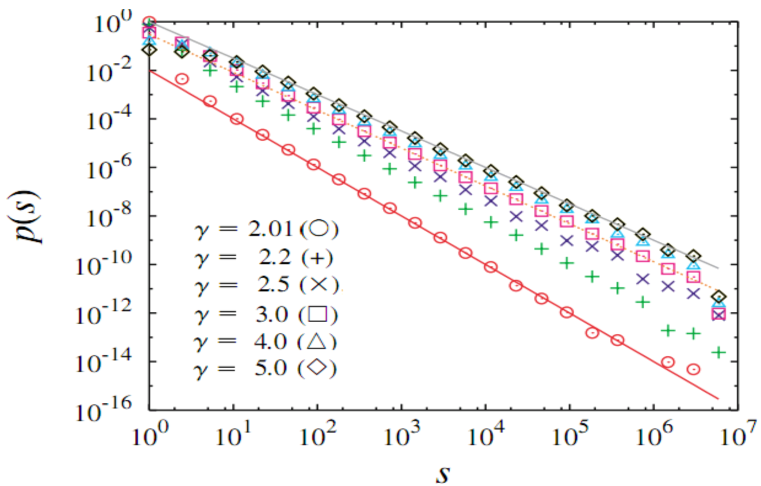
Scale-free network : $p_k \sim k^{-\gamma}$ $(2<\gamma<3)$

$$P(S) \sim S^{-\gamma/(\gamma-1)}$$

K.-I. Goh, D.-S. Lee, B. Kahng, and D. Kim, *Phys. Rev. Lett.* **91**, 148701 (2003)

BRANCHING PROCESS MODEL





Branching Process

Starting from a initial node, each node in generation t produces k number of offspring nodes in the next $t + 1$ generation, where k is selected randomly from a fixed probability distribution $q_k = p_{k-1}$.

Hypothesis

- No loops (tree structure)
- No correlation between branches

Fix <k>=1 to be critical → power law P(S)

Narrow distribution: $\langle k^2 \rangle$ converged

$$P(S) \sim S^{-3/2}$$

Fat tailed distribution: $q_k \sim k^{-\gamma}$ $(2<\gamma<3)$

$$P(S) \sim S^{-\gamma/(\gamma-1)}$$

K.-I. Goh, D.-S. Lee, B. Kahng, and D. Kim, *Phys. Rev. Lett.* **91**, 148701 (2003)

SHORT SUMMARY OF MODELS: UNIVERSALITY

Models	Networks	Exponents
Failure Propagation Model	ER	1.5
Overload Model	Complete Graph	1.5
BTW Sandpile Model	ER/SF	1.5 (ER) $\gamma/(\gamma - 1)$(SF)
Branching Process Model	ER/SF	1.5 (ER) $\gamma/(\gamma - 1)$(SF)

Universal for homogenous networks

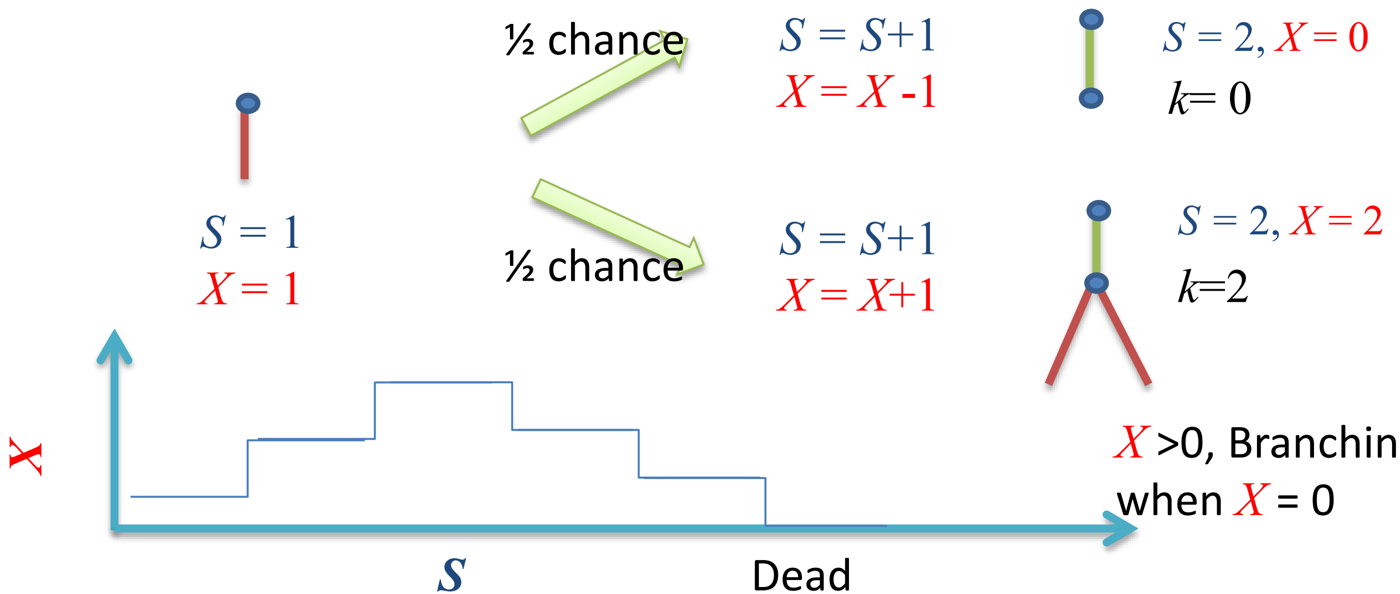
$$P(S) \sim S^{-3/2}$$

Same exponent for percolation too
(random failure, attacking, etc.)

EXPLANATION OF THE 3/2 UNIVERSALITY

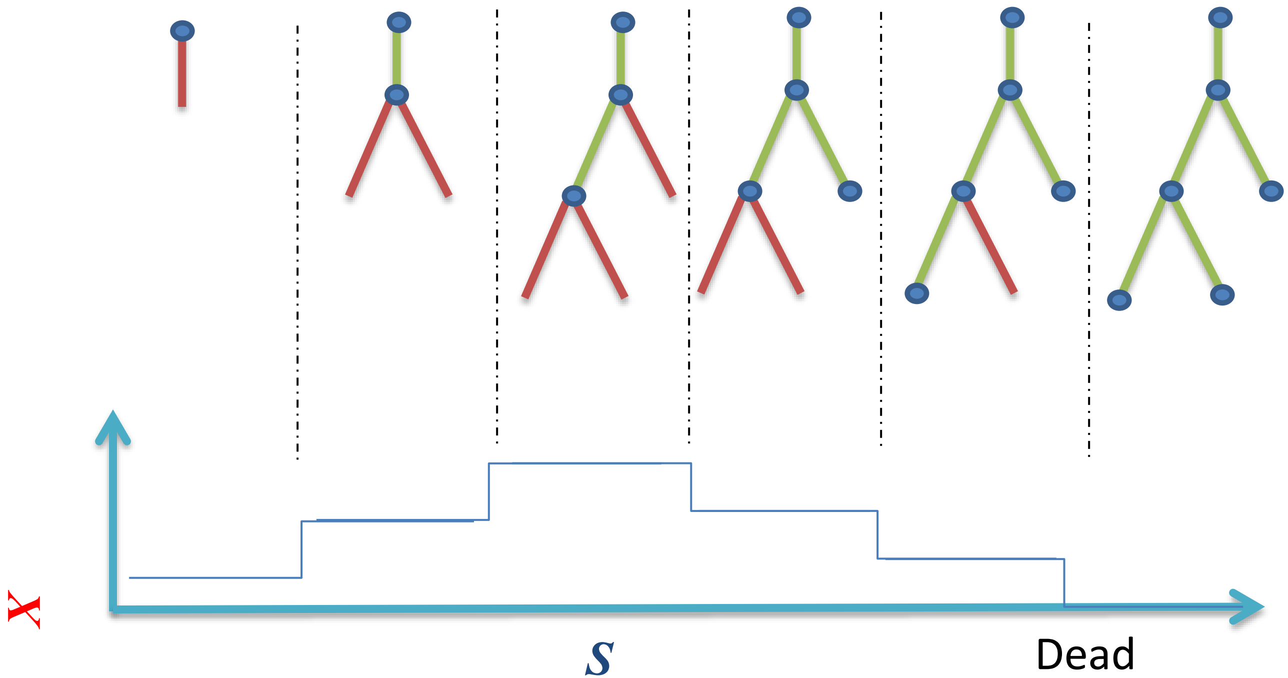
Simplest Case: $q_0 = q_2 = 1/2$, $\langle k \rangle = 1$

S: number of nodes
X: number of open branches



$X > 0$, Branching process stops when $X = 0$

EXPLANATION OF THE 3/2 UNIVERSALITY



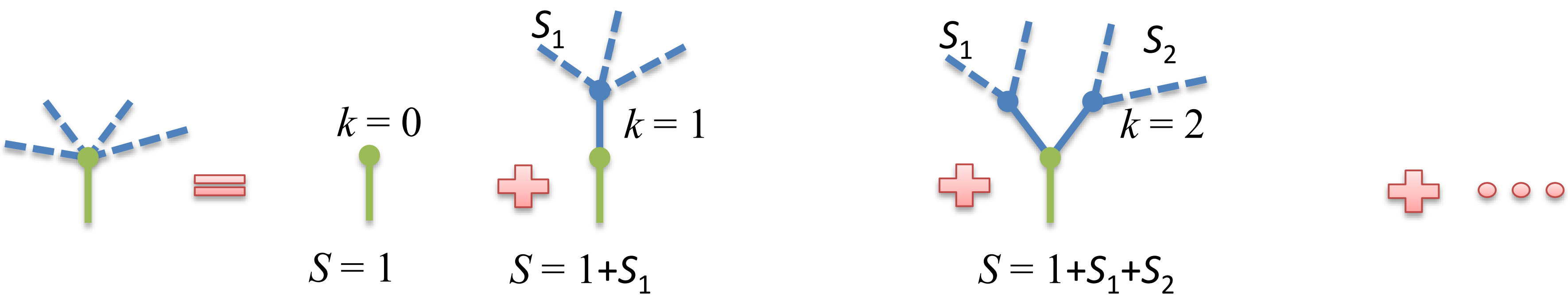
Equivalent to **1D random walk model**, where X and S are the position and time , respectively.

Question: what is the probability that $X = 0$ after S steps?

First return probability $\sim S^{-3/2}$

M. Ding, W. Yang, *Phys. Rev. E.* **52**, 207-213 (1995)

SIZE DISTRIBUTION OF BRANCHING PROCESS (CAVITY METHOD)



$$P(S) = \quad q_0\delta(1) + q_1\sum_{S_1} P(S_1)\delta(1+S_1-S) + q_2\sum_{S_1,S_2} P(S_1)P(S_2)\delta(1+S_1+S_2-S) + \cdots$$

$$P(S) = \sum_k q_k \left(\sum_{S_1,\cdots S_k} P(S_1)P(S_2)\cdots P(S_k)\delta(1+\sum_{j=1}^{k} S_j - S) \right)$$

K.-I. Goh, D.-S. Lee, B. Kahng, and D. Kim, *Physica A* **346**, 93-103 (2005)

SOLVING THE EQUATION BY GENERATING FUNCTION

Definition:

$G_S(x) = \Sigma_{S=0}\, P(S)x^S$

$G_k(x) = \Sigma_{k=0}\, q_k x^k$

Property:

$G_S(1) = G_k(1) = 1$

$G_S'(1) = \langle S \rangle,\ G_k'(1) = \langle k \rangle$

$$P(S) = \sum_k q_k \left(\sum_{S_1,\cdots S_k} P(S_1)P(S_2)\cdots P(S_k)\delta(1+\sum_{j=1}^{k} S_j - S) \right)$$

$$G_S(x) = \sum_k q_k \left(\sum_{S_1,\cdots S_k} P(S_1)\cdots P(S_k)x^{1+\sum_j S_j} \right) = \sum_k q_k x G_S(x)^k$$
$$= xG_k(G_S(x))$$

Phase Transition

$\langle S \rangle = G_S'(1) = 1 + G_k'(1)\, G_S'(1) = 1 + \langle k \rangle \langle S \rangle$, then

$\langle S \rangle = 1/(1 - \langle k \rangle)$

The average size $\langle S \rangle$ **diverges** at $\langle k \rangle_c = 1$

FINDING THE CRITICAL EXPONENT FROM EXPANSION

Definition:

$$G_S(x) = \Sigma_{S=0}\, P(S)x^S$$

$$G_k(x) = \Sigma_{k=0}\, q_k x^k$$

Theorem:

If $P(k) \sim k^{-\gamma}$ $(2<\gamma<3)$, then for $\delta x < 0$, $|\delta x| << 1$

$$G(1+\delta x) = 1 + \langle k\rangle\delta x + \langle k(k-1)/2\rangle\, (\delta x)^2 + \ldots + O(|\delta x|^{\gamma-1})$$

$$P(S) \sim S^{-\alpha}, 1 < \alpha < 2$$

$$G_S(1+\delta x) \approx 1 + A|\delta x|^{\alpha-1}$$

Homogenous case: $\langle k^2\rangle$ converged

$\langle k\rangle = 1$, $\langle k^2\rangle < \infty$

$$G_k(1+\delta x) \approx 1 + \delta x + B\delta x^2$$

Inhomogeneous case: $\langle k^2\rangle$ diverged

$\langle k\rangle = 1$, $q_k \sim k^{-\gamma}$ $(2<\gamma<3)$

$$G_k(1+\delta x) \approx 1 + \delta x + B|\delta x|^{\gamma-1}$$

CRITICAL EXPONENT FOR HOMOGENOUS CASE

Homogenous case

$$G_k(1+\delta x) \approx 1 + \delta x + B\delta x^2$$
$$G_S(1+\delta x) \approx 1 + A|\delta x|^{\alpha-1}$$

$$G_S(x) = xG_k(G_S(x))$$

$$G_S(x) \approx 1 + A|\delta x|^{\alpha-1}$$
$$\begin{aligned} xG_k(G_S(x)) &\approx (1+\delta x)[1+(G_S(1+\delta x)-1) + B(G_S(1+\delta x)-1)^2] \\ &\approx (1+\delta x)[1+A|\delta x|^{\alpha-1} + AB|\delta x|^{2\alpha-2}] \\ &= 1 + A|\delta x|^{\alpha-1} + AB|\delta x|^{2\alpha-2} + \delta x + O(|\delta x|^{\alpha}) \end{aligned}$$

The lowest order reads $AB|\delta x|^{2\alpha-2} + \delta x = 0$, which requires $2\alpha - 2 = 1$ and $A = 1/B$. Or,

$$\alpha = 3/2$$

CRITICAL EXPONENT FOR INHOMOGENEOUS CASE

Inhomogeneous case

$$G_k(1+\delta x) \approx 1 + \delta x + B|\delta x|^{\gamma - 1}$$
$$G_S(1+\delta x) \approx 1 + A|\delta x|^{\alpha - 1}$$

$$G_S(x) = xG_k(G_S(x))$$

$$G_S(x) \approx 1 + A|\delta x|^{\alpha - 1}$$

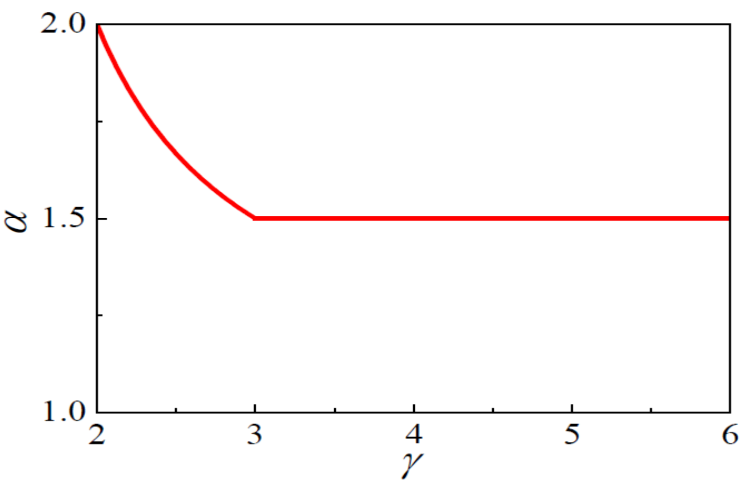
$$\begin{aligned} xG_k(G_S(x)) &\approx (1+\delta x)[1 + (G_S(1+\delta x)-1) + B\,|G_S(1+\delta x)-1|^{\gamma - 1}] \\ &\approx (1+\delta x)[1 + A|\delta x|^{\alpha - 1} + AB|\delta x|^{(\alpha - 1)(\gamma - 1)}] \\ &= 1 + A|\delta x|^{\alpha - 1} + AB|\delta x|^{(\alpha - 1)(\gamma - 1)} + \delta x + O(|\delta x|^{\alpha}) \end{aligned}$$

The lowest order reads $AB|\delta x|^{(\alpha - 1)(\gamma - 1)} + \delta x = 0$, which requires $(\alpha - 1)(\gamma - 1) = 1$ and $A = 1/B$. Or,

$$\alpha = \gamma/(\gamma - 1)$$

COMPARING THE PREDICTION WITH THE REAL DATA

$$P(S) \sim S^{-\alpha}, \alpha = \begin{cases} 3/2, & \gamma > 3 \\ \gamma/(\gamma-1), & 2 < \gamma < 3 \end{cases}$$



Blackout

Source	Exponent	Quantity
North America	2.0	Power
Sweden	1.6	Energy
Norway	1.7	Power
New Zealand	1.6	Energy
China	1.8	Energy

Earthquake $\alpha \approx 1.67$

I. Dobson, B. A. Carreras, V. E. Lynch, D. E. Newman, *CHAOS* **17**, 026103 (2007)

Y. Y. Kagan, *Phys. Earth Planet. Inter.* **135** (2–3), 173–209 (2003)