Network Epidemics

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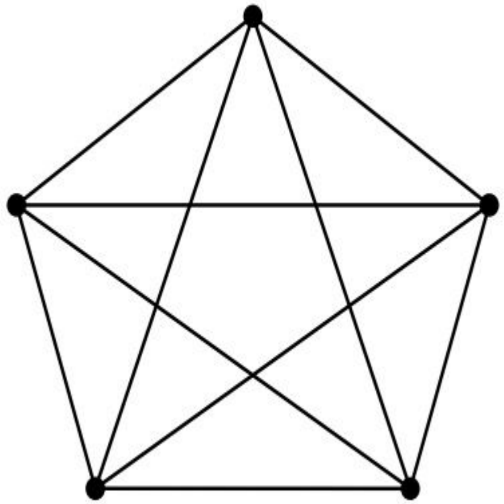
Outline

- ❑ **Setup**
- ❑ Susceptible-Infected (SI) Model
- ❑ Susceptible-Infected-Susceptible (SIS) Model
- ❑ Summary

Previous Assumptions

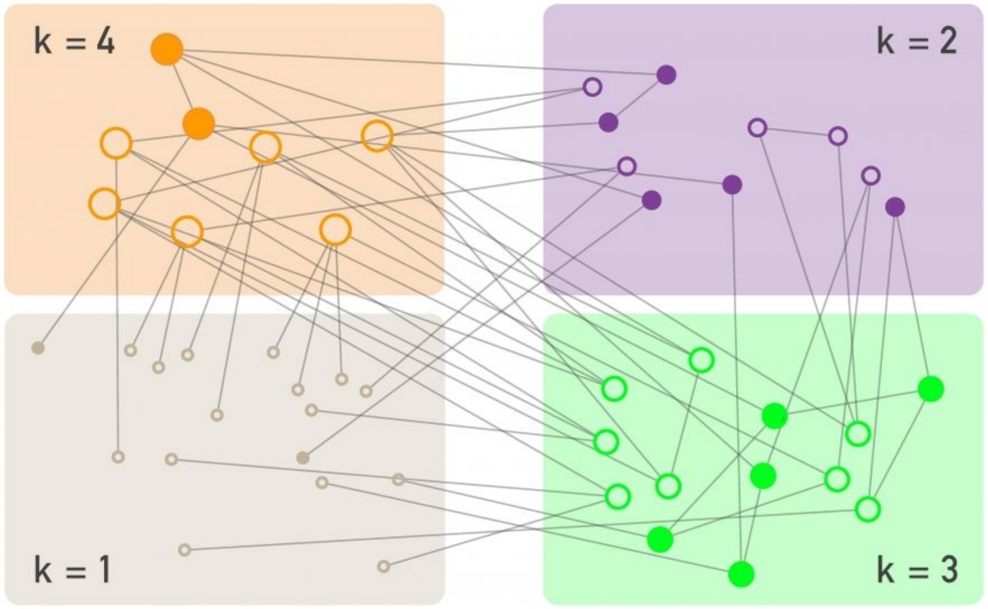
- ❑ Homogenous mixing
- ❑ Nodes have similar ⟨k⟩



We will neglect these going forward...

Degree Block Approximation

- Nodes with higher k are more likely to get infected
- Need to distinguish nodes based on degree



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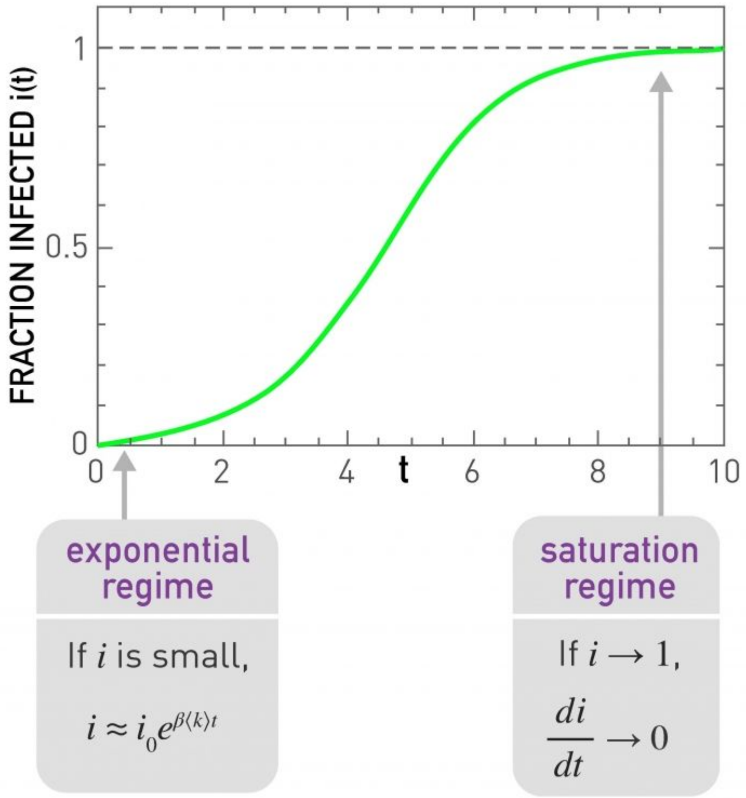
Review of the SI Model

- ❑ Susceptible ➔ Infected
- ❑ Every node ends up infected



Implementing Degree Block Approximation

$$\frac{di}{dt} = \beta \langle k \rangle si = \beta \langle k \rangle i(1-i) \quad \rightarrow \quad \frac{di_k}{dt} = \beta(1-i_k)k\Theta_k$$

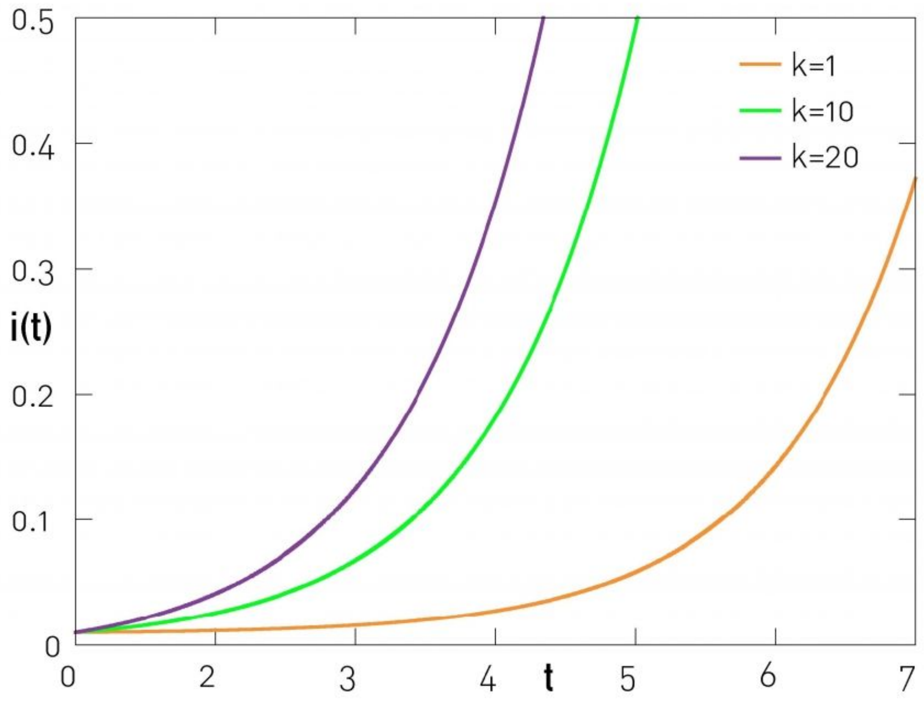
$$i = \frac{i_0 e^{\beta\langle k\rangle t}}{1-i_0+i_0 e^{\beta\langle k\rangle t}} \quad \rightarrow \quad i_k = i_0\left(1+\frac{k(\langle k\rangle-1)}{\langle k^2\rangle-\langle k\rangle}\left(e^{t/\tau^{SI}}-1\right)\right)$$

$$\tau = \frac{1}{\beta\langle k\rangle} \quad \rightarrow \quad \tau^{SI} = \frac{\langle k\rangle}{\beta(\langle k^2\rangle-\langle k\rangle)}$$

Infected

- The higher the degree, the faster to be infected

$$i_k = i_0 \left(1 + \frac{k(\langle k\rangle - 1)}{\langle k^2\rangle - \langle k\rangle}\left(e^{t/\tau^{SI}} - 1\right)\right)$$



Characteristic Time

- In General

$$\tau^{SI} = \frac{\langle k \rangle}{\beta(\langle k^2 \rangle - \langle k \rangle)}$$

- Random Network → $\langle k^2 \rangle = \langle k \rangle(\langle k \rangle + 1)$

$$\tau_{ER}^{SI} = \frac{1}{\beta \langle k \rangle}$$

- Scale-free Network with $\gamma \geq 3$
  - $\langle k \rangle$ and $\langle k^2 \rangle$ are finite

$$\tau \cong \tau_{ER}^{SI} = \frac{1}{\beta \langle k \rangle}$$

- Scale-free Network with $\gamma \leq 3$
  - As N → ∞, then $\langle k^2 \rangle \rightarrow \infty$

$$\tau^{SI} \rightarrow 0$$

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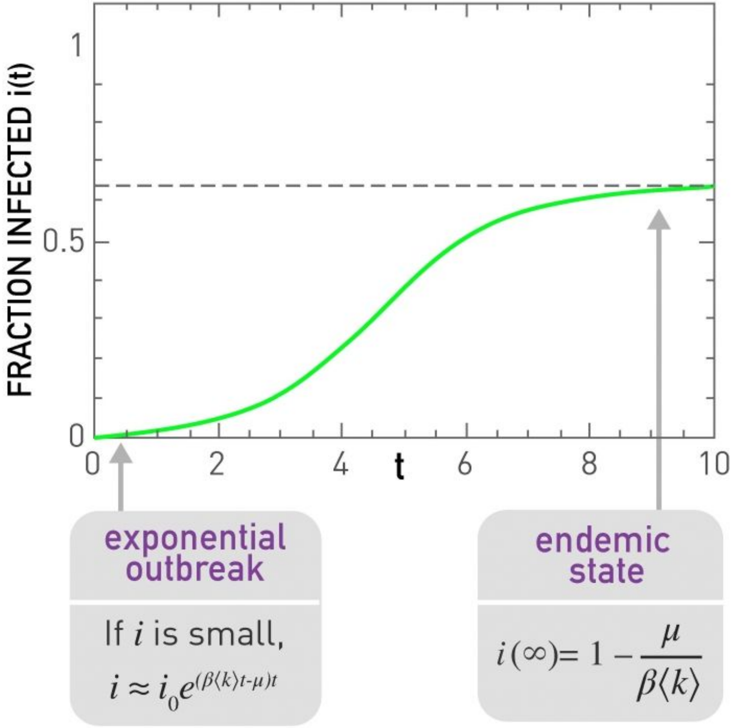
Review of the SIS Model

- ❑ Susceptible → Infected
- ❑ Infected → Susceptible
- ❑ Endemic state or disease dies out



Implementing Degree Block Approximation

$$\frac{di}{dt} = \beta \langle k \rangle i(1-i) - \mu i \quad \longrightarrow \quad \frac{di_k}{dt} = \beta(1-i_k)k\Theta_k(t) - \mu i_k$$

$$\tau = \frac{1}{\mu(R_0 - 1)} \quad \longrightarrow \quad \tau^{SIS} = \frac{\langle k \rangle}{\beta\langle k^2 \rangle - \mu\langle k \rangle}$$

Spreading Rate $\longrightarrow$ $\lambda = \frac{\beta}{\mu}$

Random Network

Random Network ➔ $\langle k^2 \rangle = \langle k \rangle (\langle k \rangle + 1)$

$$\tau^{SIS} = \frac{\langle k \rangle}{\beta \langle k^2 \rangle - \mu \langle k \rangle} \longrightarrow \tau_{ER}^{SIS} = \frac{1}{\beta(\langle k \rangle + 1) - \mu}$$

Epidemic Threshold

$$\tau_{ER}^{SIS} = \frac{1}{\beta(\langle k \rangle + 1) - \mu} + \lambda = \frac{\beta}{\mu} \longrightarrow \lambda_c = \frac{1}{\langle k \rangle + 1}$$

Scale-Free Network

Epidemic Threshold

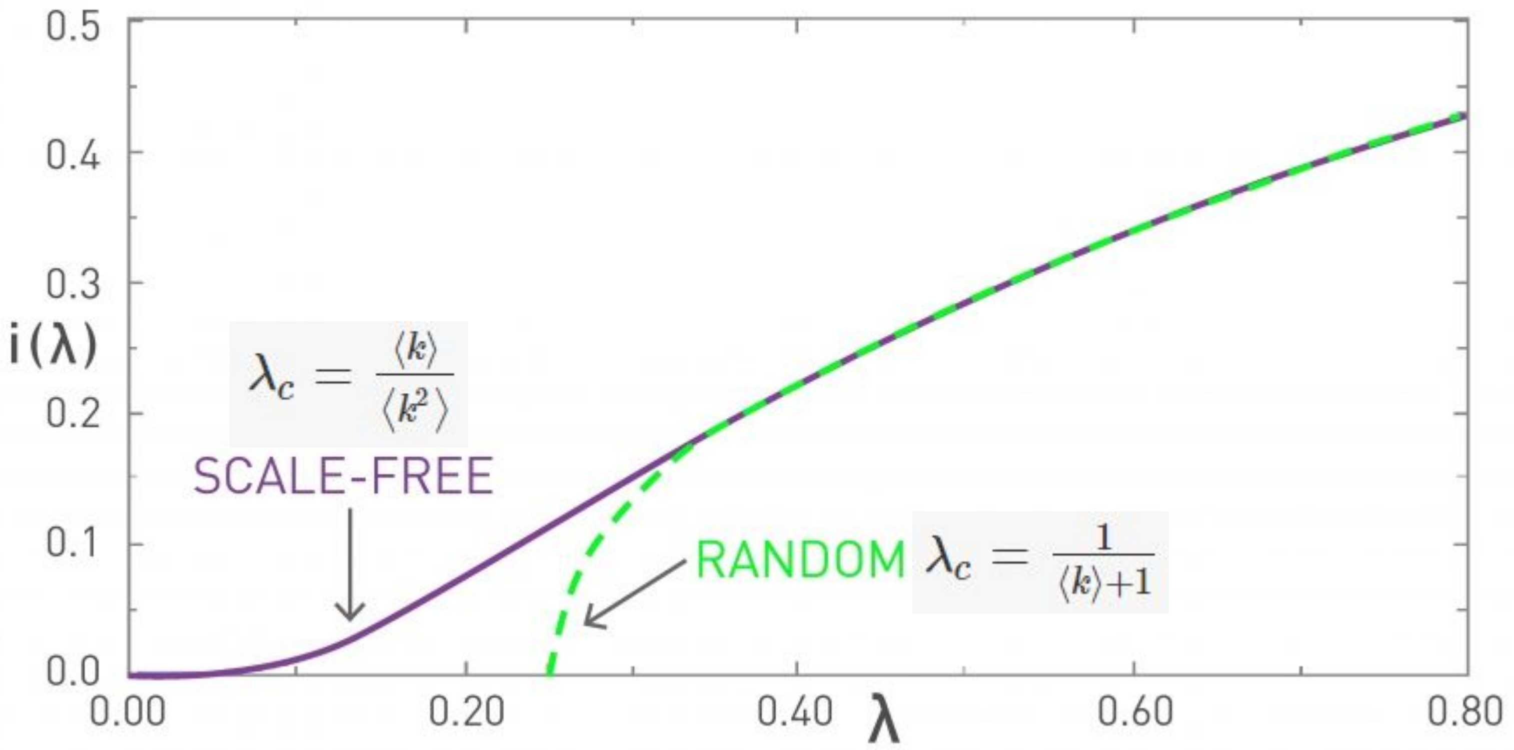
$$\tau^{SIS} = \frac{\langle k \rangle}{\beta \langle k^2 \rangle - \mu \langle k \rangle} + \lambda = \frac{\beta}{\mu}$$

$$\lambda_c = \frac{\langle k \rangle}{\langle k^2 \rangle}$$

Scale-free, therefore as N → ∞ then ⟨k²⟩ → ∞

$2 < \gamma < 3$	$\lambda_c = 0$ $\Theta(\lambda) \sim (k_{\min}\lambda)^{(\gamma-2)/(3-\gamma)}$ $i(\lambda) \sim \lambda^{1/(3-\gamma)}$
$\gamma = 3$	$\lambda_c = 0$ $\Theta(\lambda) \approx \frac{e^{-1/k_{\min}\lambda}}{\lambda k_{\min}}(1 - e^{-1/k_{\min}\lambda})^{-1}$ $i(\lambda) \sim 2e^{-1/k_{\min}\lambda}$
$3 < \gamma < 4$	$\lambda_c > 0$ $i(\lambda) \sim \left(\lambda - \frac{\gamma-3}{k_{\min}(\gamma-2)}\right)^{1/(\gamma-3)}$
$\gamma > 4$	$\lambda_c > 0$ $i(\lambda) \sim \lambda - \frac{\gamma-3}{k_{\min}(\gamma-2)}$

Random vs Scale-Free



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Summary

- ❑ Hubs are far more important than originally thought

Model	Continuum Equation	τ	λ_c
SI	$\frac{di_k}{dt} = \beta [1 - i_k] k\theta_k$	$\frac{\langle k \rangle}{\beta\left(\left\langle k^2 \right\rangle - \langle k \rangle\right)}$	0
SIS	$\frac{di_k}{dt} = \beta [1 - i_k] k\theta_k - \mu i_k$	$\frac{\langle k \rangle}{\beta\left\langle k^2 \right\rangle - \mu \langle k \rangle}$	$\frac{\langle k \rangle}{\left\langle k^2 \right\rangle}$
SIR	$\frac{di_k}{dt} = \beta s_k \theta_k - \mu i_k$ $s_k = 1 - i_l - r_k$	$\frac{\langle k \rangle}{\beta\left\langle k^2 \right\rangle - (\mu + \beta)\langle k \rangle}$	$\frac{1}{\frac{\left\langle k^2 \right\rangle}{\langle k \rangle} - 1}$

Questions?

References

Barabási, A.-L. (2016). *Network science*. Cambridge University Press.