Phys. Rev. Lett. 105, 068701 (2010) [4 pages]

Network Synchronization in a Noisy Environment with Time Delays: Fundamental Limits and Trade-Offs

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(Dated: June 30, 2010)

Abstract

We study the effects of nonzero time delays in stochastic synchronization problems with linear couplings in an arbitrary network. Using the known exact threshold value from the theory of differential equations with delays, we provide the synchronizability threshold for an arbitrary network. Further, by constructing the scaling theory of the underlying fluctuations, we establish the absolute limit of synchronization efficiency in a noisy environment with uniform time delays, i.e., the minimum attainable value of the width of the synchronization landscape. Our results have also strong implications for optimization and trade-offs in network synchronization with delays.

PACS numbers: 89.75.Hc, 05.40.-a, 89.20.Ff

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In network synchronization problems [1], individual units, represented by nodes in the network, attempt to adjust their local state variables (e.g., pace, load, orientation) in a decentralized fashion. They interact or communicate only with their local neighbors in the network, often with the intention to improve global performance. These interactions or couplings can be represented by directed or undirected, weighted or unweighted links. Applications of the corresponding models range from physics, biology, computer science to control theory, including synchronization problems in distributed computing [2], coordination and control in communication networks [3–7], flocking animals [8, 9], bursting neurons [10–13], and cooperative control of vehicle formation [14].

There has been a massive amount of research focusing on the efficiency and optimization of synchronization problems [1, 15–18] in various complex network topologies, including weighted [3, 19] and directed [6, 20–22] networks. In this Letter, we study an aspect of stochastic synchronization problems which is present in all real communication, information, and computing networks [5–7, 23, 24], including neurobiological networks [12, 13]: the impact of *time delays* on synchronizability and on the breakdown of synchronization. The presence of time delays, however, will also present possible scenarios for trade-offs. Here we show that when synchronization networks are stressed by large delays, reducing local coordination effort will actually improve global coordination. Similarly subtle results have also been found in neurobiological networks with the synchronization efficiency exhibiting non-monotonic behavior as a function of the delay [12, 13].

For our study, we consider the simplest stochastic model with linear local relaxation, where network-connected agents locally adjust their state to closely match that of their neighbors (e.g., load, or task allocation) in an attempt to improve global performance. However, they react to the information or signal received from their neighbors with some time lag (as result of finite processing, queueing, or transmission delays), motivating our study of the coupled stochastic equations of motion with delay,

$$\partial_t h_i(t) = -\sum_{j=1}^{N} C_{ij}[h_i(t-\tau_{ij}) - h_j(t-\tau_{ij})] + \eta_i(t) \ . \tag{1}$$

Here, $h_i(t)$ is the generalized local state variable on node i and $\eta_i(t)$ is a delta-correlated noise with zero mean and variance $\langle\eta_i(t)\eta_j(t')\rangle = 2D\delta_{ij}\delta(t-t')$, where D is the noise intensity. $C_{ij}=C_{ji}\geq 0$ is the symmetric coupling strength ($C_{ij}=W_{ij}A_{ij}$ in general weighted networks, where A_{ij} is the adjacency matrix and W_{ij} is the link weight). $\tau_{ij}>0$ is the time delay between two connected nodes i and j. For initial conditions we use $h_i(t)$=0 for $t\leq 0$. Eq. (1) is also referred to as the Edwards-Wilkinson process [25] on networks [3] with time-delays. Without the noise term, the above equation is often referred to as the consensus problem [5, 6] on the respective network.

The standard observable in stochastic synchronization problems, where relaxation competes with noise, is the width of the synchronization landscape [2, 3, 17, 18]

$$\langle w^2(t)\rangle \equiv \left\langle \frac{1}{N}\sum_{i=1}^{N}[h_i(t)-\bar{h}(t)]^2 \right\rangle \;, \tag{2}$$

where $\bar{h}(t) = (1/N)\sum_{i=1}^{N} h_i(t)$ is the global average of the local state variables and $\langle \ldots \rangle$ denotes an ensemble average over the noise. A network of N nodes is synchronizable if $\langle w^2(\infty)\rangle<\infty$, i.e., if the width approaches a finite value in the $t\to\infty$ limit. The smaller the width, the better the synchronization.

In the case of uniform delays $\tau_{ij} \equiv \tau$, the focus of this Letter, one can rewrite Eq. (1) as

$$\partial_t h_i(t) = -\sum_{j=1}^{N}\Gamma_{ij}h_j(t-\tau) + \eta_i(t) \;, \tag{3}$$

where $\Gamma_{ij} = \delta_{ij}\sum_l C_{il} - C_{ij}$, is the symmetric network Laplacian. In this case, by diagonalizing the network Laplacian, one can decompose the problem into N *independent* modes

$$\partial_t \tilde{h}_k(t) = -\lambda_k \tilde{h}_k(t-\tau) + \tilde{\eta}_k(t) \;, \tag{4}$$

where λ_k, $k = 0, 1, 2, \ldots, N-1$, are the eigenvalues of the network Laplacian and $\langle\tilde{\eta}_k(t)\tilde{\eta}_l(t')\rangle$=$2D\delta_{kl}\delta(t-t')$. For a single-component (or connected) network, the Laplacian has a single zero mode (indexed by k=0) with λ_0=0, while λ_k>0 for $k\geq 1$. Using the above eigenmode decomposition, the width of the synchronization landscape can be expressed as $\langle w^2(t)\rangle = (1/N)\sum_{k=1}^{N-1}\langle\tilde{h}_k^2(t)\rangle$ [3].

For example, for *zero time delay* (τ=0), one immediately finds $\langle w^2(t)\rangle$=$(1/N)\sum_{k=1}^{N-1} D\lambda_k^{-1}(1-e^{-2\lambda_k t})$. The above expression explicitly shows that every finite connected network with zero time delay is synchronizable, $\langle w^2(\infty)\rangle<\infty$. In the limit of infinite network size, however, network ensembles with a vanishing (Laplacian) spectral gap may become unsynchronizable, depending on the details of the small-λ behavior of the density of eigenvalues [1–3].

In the case of *non-zero uniform delays*, the case considered here, the eigenmodes of the problem are again governed by a stochastic equation of motion of identical form for all $k\geq 1$

[Eq. (4)]. Thus, understanding the time-evolution of a single stochastic variable $\tilde{h}_k(t)$ and its fluctuations will provide both full insight to the synchronizability condition of the network-coupled system and a framework to compute the width of the synchronization landscape. Therefore, to ease notational burden and to direct our focus to a single stochastic variable, we will temporarily drop the index k referring to a specific eigenmode, and study the stochastic differential equation

$$\partial_t \tilde{h}(t) = -\lambda \tilde{h}(t-\tau) + \tilde{\eta}(t) \tag{5}$$

with $\langle \tilde{\eta}(t)\tilde{\eta}(t')\rangle = 2D\delta(t-t')$. Using standard Laplace transformation with initial conditions $\tilde{h}(t)=0$ for $t \leq 0$, one finds

$$\tilde{h}(t) = \int_0^t dt' \tilde{\eta}(t') \sum_\alpha \frac{e^{s_\alpha (t-t')}}{1+\tau s_\alpha} \ , \tag{6}$$

where s_α, $\alpha = 1, 2, \ldots$, are the solutions of the characteristic equation

$$s + \lambda e^{-\tau s} = 0 \tag{7}$$

in the complex plane. The above complex equation has an *infinite* number of solutions for $\tau > 0$ [5, 26, 27]. Using Eq. (6), we can write an expression for the fluctuations, averaged over the noise,

$$\langle \tilde{h}^2(t)\rangle = \sum_{\alpha,\beta} \frac{-2D(1-e^{(s_\alpha+s_\beta)t})}{(1+\tau s_\alpha)(1+\tau s_\beta)(s_\alpha+s_\beta)} \ . \tag{8}$$

The solution of Eq. (7) with the largest real part governs the long-time temporal behavior of the respective mode (e.g., stability, approach to, or relaxation in the steady state). The condition for $\langle \tilde{h}^2(\infty)\rangle$ to remain finite is $\mathrm{Re}(s_\alpha) < 0$ for all α. As has been shown for Eq. (7), this inequality holds if $\tau\lambda < \pi/2$ [5, 26, 27]. In Fig. 1 we show the time-dependent width of the fluctuations associated with a single stochastic variable, obtained by numerically integrating Eq. (5) for a few characteristic cases [28].

Returning to the context of network synchronization, synchronizability requires a finite steady-state width, $\langle w^2(\infty)\rangle = (1/N)\sum_{k=1}^{N-1}\langle \tilde{h}_k^2(\infty)\rangle < \infty$. Thus, for uniform time delays in a given network, each and every $k \geq 1$ mode must have finite steady-state fluctuations $\langle \tilde{h}_k^2(\infty)\rangle < \infty$. This implies that one must have $\tau\lambda_k < \pi/2$ for *all* $k \geq 1$ modes, or equivalently [29],

$$\tau\lambda_{\max} < \pi/2 \ . \tag{9}$$

The above exact delay threshold for synchronizability has some immediate and profound

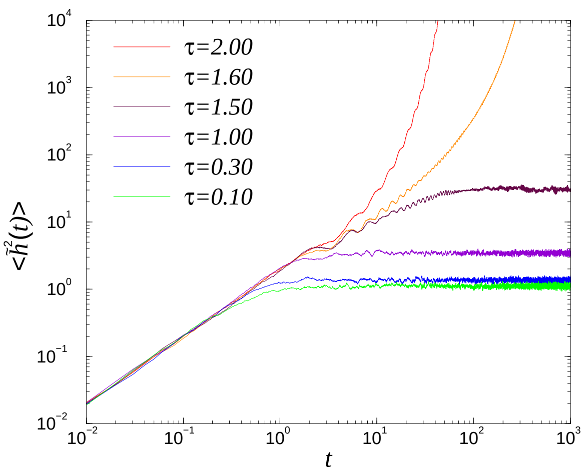

FIG. 1: Time series $\langle\tilde{h}^2(t)\rangle$ for different delays, obtained by numerically integrating Eq. (5) and averaging over 1,000 independent realization of the noise. Here, λ=1, D=1, and Δt=0.01. The theoretical (continuum-time) threshold value of the delay (for $\langle\tilde{h}(\infty)\rangle$ to remain bounded) is τ_c=$\pi/2$ [28].

consequences for unweighted networks. Here, the coupling matrix is identical to the adjacency matrix, C_{ij}=A_{ij}, and the bounds and the scaling properties of the extreme eigenvalues of the network Laplacian are well known. In particular, $Nk_{\rm max}/(N-1)\leq\lambda_{\rm max}\leq 2k_{\rm max}$ [30, 31], where $k_{\rm max}$ is the maximum node degree in the network (i.e., $\langle\lambda_{\rm max}\rangle$=$\mathcal{O}(\langle k_{\rm max}\rangle)$). Thus, $\tau k_{\rm max}<\pi/4$ is sufficient for synchronizibility, while $\tau k_{\rm max}>\pi/2$ leads to the breakdown of synchronization with certainty. These inequalities imply that networks with potentially large degrees, e.g., scale-free (SF) networks [32, 33], are rather vulnerable to intrinsic network delays [5, 6]. For example, SF network ensembles with a natural degree cut-off exhibit $\langle\lambda_{\rm max}\rangle\sim\langle k_{\rm max}\rangle\sim N^{1/(\gamma-1)}$ for $N\gg 1$ (when the average degree $\langle k\rangle$ is held fixed), where γ is the exponent governing the power-law tail of the degree distribution [34, 35]. In turn, the probability that a realization of a random SF network ensemble of N nodes is synchronizable approaches zero for *any nonzero* delay τ in the limit of $\tau N^{1/(\gamma-1)}\gg 1$ [36].

Next, we analyze the steady-state behavior of the width in the synchronizable regime. We accomplish this by investigating the basic scaling features of the steady-state fluctuations of a single stochastic variable, $\langle\tilde{h}^2(\infty)\rangle$, governed by Eq. (5), which can be associated with an arbitrary mode. In this regime one must have Re(s_α)<0 for all α, or equivalently, $\tau\lambda<\pi/2$. Defining a new variable $z\equiv\tau s$, Eq. (7) can be rewritten $z+\lambda\tau e^{-z}=0$, i.e., for a given λ and τ, the solutions for the scaled variable z can only depend on $\lambda\tau$, z_α=$z_\alpha(\lambda\tau)$, $\alpha=1,2,\ldots$.

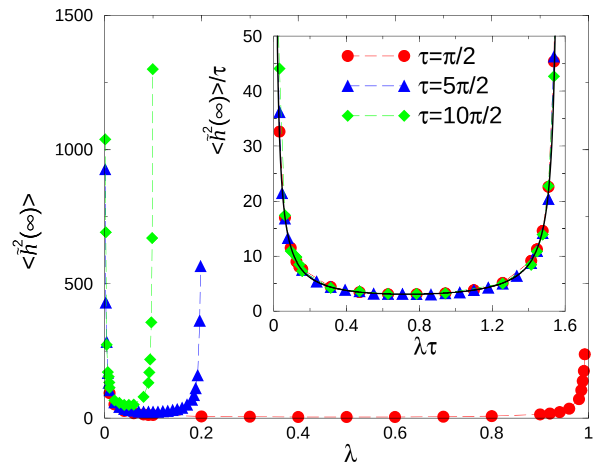

FIG. 2: Steady-state fluctuations $\langle\tilde{h}(\infty)\rangle$ obtained by numerically integrating Eq. (5) as a function of λ for different τ values. Here, D=1, and Δt=0.01. The inset shows the scaled plot of the same data points, $\langle\tilde{h}^2(\infty)\rangle/\tau$ vs $\lambda\tau$, together with the numerically fitted scaling function $f(x)$ (solid curve) [28, 36].

Thus, the solutions of the characteristic equation Eq. (7) must exhibit the scaling form $s_\alpha = \tau^{-1} z_\alpha(\lambda\tau)$, $\alpha = 1, 2, \ldots$. Substituting the above expression into Eq. (8) and taking the $t\to\infty$ limit immediately yields the scaling form

$$\langle\tilde{h}^2(\infty)\rangle = D\tau f(\lambda\tau) \,. \tag{10}$$

Thus, for a single stochastic variable $\tilde{h}(t)$ governed by the stochastic differential equation Eq. (5) (simple relaxation in a noisy environment with delay), plotting $\langle\tilde{h}^2(\infty)\rangle/\tau$ vs $\lambda\tau$ (for a fixed noise intensity D) should yield full data collapse, as demonstrated in Fig. 2 [28]. [While we do not have an analytic expression for the scaling function, for small arguments it asymptotically has to scale as $f(x)\simeq 1/x$ to reproduce the exact limiting case of zero delay, $\langle\tilde{h}^2(\infty)\rangle\simeq D/\lambda$. Further, we numerically found that in the vicinity of $\pi/2$, it approximately diverges as $(\pi/2 - x)^{-1}$.] The scaling function $f(x)$ is clearly non-monotonic; it exhibits a single minimum, at approximately $x^*\approx 0.73$ with $f^*=f(x^*)\approx 3.1$. The immediate message of the above result is rather interesting: For a single stochastic variable governed by Eq. (5) with a nonzero delay, there is an optimal value of the relaxation coefficient $\lambda^*=x^*/\tau$, at which point the steady-state fluctuations attain their minimum value $\langle\tilde{h}^2(\infty)\rangle=D\tau f^*\approx 3.1D\tau$. This is in stark contrast with the zero-delay case where $\langle\tilde{h}^2(\infty)\rangle=D/\lambda$, i.e., there the steady-state fluctuation is a monotonically decreasing function of the relaxation coefficient.

In addition to gaining fundamental insights, constructing the scaling function $f(x)$ numerically with some acceptable precision of the single variable problem [36] (Fig. 2 inset) also provides a method to obtain the steady-state width of the network-coupled system: one can numerically diagonalize the Laplacian of the underlying network and employ the scaling function $f(x)$ to obtain the width,

$$\langle w^2(\infty)\rangle = \frac{1}{N}\sum_{k=1}^{N-1}\langle \tilde{h}_k^2(\infty)\rangle = \frac{D\tau}{N}\sum_{k=1}^{N-1} f(\lambda_k\tau)\ . \tag{11}$$

Further, we can now extract the minimum attainable width of the synchronization landscape in a noisy environment with uniform time delays. For a fixed τ, each term in Eq. (11) can be minimized by choosing $\lambda_k = x^*/\tau$ for all $k{\geq}1$. Then

$$\langle w^2(\infty)\rangle^* = \frac{N-1}{N} D\tau f^* \approx 3.1 D\tau \tag{12}$$

for large N. This number, the fundamental limit of synchronization efficiency in a noisy environment with uniform time delays, can be used as a base-line value when comparing networks from the viewpoint of synchronization efficiency. Note that there is a trivial network which realizes the optimal behavior: the fully connected graph with identical coupling constants $C_{ij}{=}x^*/N\tau$ for all $i{\neq}j$. (This network has $N{-}1$ identical non-zero eigenvalues, $\lambda_k{=}x^*/\tau$ for all $k{\geq}1$.) In general, networks with a narrow spectrum centered about $\lambda^* = x^*/\tau$ shall perform closer to optimal. How to construct such networks with possible topological and cost constraints is a different and challenging question which we will not pursue in detail here, but we note that essentially the same problem arises in the broader context of synchronization of generalized dynamical systems [20, 21]. Recent methods tackling this issue involve locally reweighting and/or removing links from the networks to achieve optimal performance [21].

The essential non-monotonic feature of the scaling function $f(x)$ in Eq. (11) (including the potentially diverging contributions from large eigenvalues beyond the threshold) presents various trade-off scenarios in network synchronization problems with delays. As the simplest and obvious application of the above results, consider a network which is stressed by large delays beyond its threshold, $\tau\lambda_{\max}{>}\pi/2$ (so that the largest fluctuations and the width are growing exponentially without bound). Then even a suitably chosen uniform reduction of all couplings $C'_{ij}{=}pC_{ij}$ ($\lambda'_k{=}p\lambda_k$) with $p{<}\pi/2\lambda_{\max}\tau$ will lead to the stabilization of the system, with a finite steady-state width. In communication and computing networks, the effective coupling strength C_{ij} can be controlled by the frequency (or rate) of local synchronizations through the respective link [2]. The above results then suggest that when the

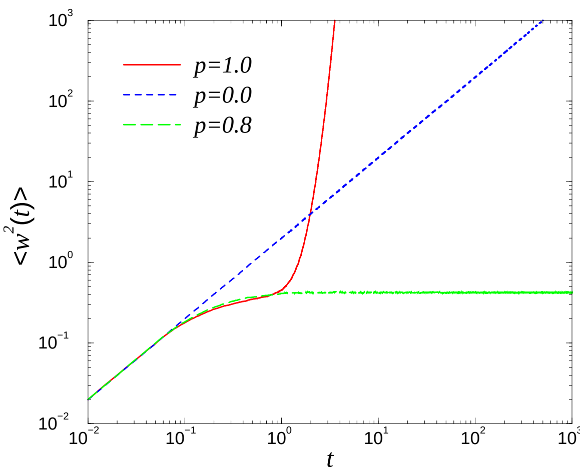

FIG. 3: Time dependent width for τ=1.2$\pi/2\lambda_{\max}$ for different values of the local synchronization rate p on a fixed graph, obtained by the numerical integration of Eq. (1) with Δt=0.005 and D=1.0. The underlying network is a Barabási-Albert SF graph [32] with N=100, $\langle k\rangle\approx 6$, and $\lambda_{\max}\approx 32$.

system is beyond its stability threshold, synchronizing sufficiently *less frequently*, can lead to stabilization and better coordination [36]. Figure 3 shows results for the case when the communication neighborhood is fixed, but the local synchronizations through the links [the coupling terms in Eq. (1)] are only performed with probability $p\leq 1$, while invoking the noise term at every time step [37]. Indeed, reducing the local synchronization rate can improve global performance. In fact, even performing no local synchronizations at all (p=0) leads to a slower power-law divergence of the width with time, $\langle w^2(t)\rangle \simeq 2Dt$, as opposed to the exponential divergence governed by the largest eigenvalue(s) above the threshold.

In summary, we have obtained the delay threshold for the simplest stochastic synchronization problem with linear couplings in an arbitrary network. Further, exploring and investigating the scaling properties of the fluctuations associated with the eigenmodes of the network Laplacian, we found the minimum attainable steady-state width of the synchronization landscape in any network. The non-monotonic feature of the scaling function governing the fluctuations can guide potential trade-offs and optimization in network synchronization. For systems with more general (non-linear) node dynamics, one can also expect that the synhronizability/stability phase diagram will exhibit non-monotonic behavior as a function of the coupling strength and/or the delays [12, 13, 23, 24]. In real communication and information networks, the delays τ_{ij} are not uniform [4, 7], but are affected by the network neighborhood and spatial distance. We currently investigate the impact of heterogeneous delays on network synchronization [36].

We thank S. Sreenivasan and T. Caraco for discussions and a careful reading of the manuscript. This work was supported in part by DTRA Award No. HDTRA1-09-1-0049 and by the Army Research Laboratory under Cooperative Agreement Number W911NF-09-2-0053. The views and conclusions contained in this document are those of the authors and should not be interpreted as representing the official policies, either expressed or implied, of the Army Research Laboratory or the U.S. Government.

[1] A. Arenas, A. Díaz-Guilera, J. Kurths, Y. Moreno, and C. Zhou, Phys. Rep.**469**, 93 (2008).

[2] G. Korniss, M.A. Novotny, H. Guclu, and Z. Toroczkai, P.A. Rikvold, Science **299**, 677 (2003).

[3] G. Korniss, Phys. Rev. E **75**, 051121 (2007).

[4] R. Johari and D. Kim Hong Tan, IEEE/ACM Trans. Networking **9**, 818 (2001).

[5] R. Olfati-Saber and R.M. Murray, IEEE Trans. Automat. Contr. **49**, 1520 (2004).

[6] R. Olfati-Saber, J.A. Fax, and R.M. Murray, Proc. IEEE **95**, 215 (2007).

[7] G. Scutari, S. Barbarossa, and L. Pescosolido, IEEE Trans. Sig. Process. **56**, 1667 (2008).

[8] T. Vicsek *et al.*, Phys. Rev. Lett. **75**, 1226 (1995).

[9] F. Cucker and S. Smale, IEEE Trans. Automat. Contr. **52**, 852 (2007).

[10] E. Izhikevich, SIAM Rev. **43**, 315 (2001).

[11] F.M. Atay, Phys. Rev. Lett. **91**, 094101 (2003).

[12] Q. Wang, Z. Duan, M. Perc, and G. Chen, Europhys. Lett. **83**, 50008 (2008).

[13] Q. Wang, M. Perc, Z. Duan, and G. Chen, Phys. Rev. E **80**, 026206 (2009).

[14] J.A. Fax and R.M. Murray, IEEE Trans. Automat. Contr. **49**, (2004).

[15] M. Barahona and L.M. Pecora, Phys. Rev. Lett. **89**, 054101 (2002).

[16] T. Nishikawa, A.E. Motter, Y.-C. Lai, and F.C. Hoppensteadt, Phys. Rev. Lett. **91**, 014101 (2003).

[17] C. E. La Rocca, L. A. Braunstein, and P. A. Macri, Phys. Rev. E **77**, 046120 (2008).

[18] C. E. La Rocca, L. A. Braunstein, and P. A. Macri, Phys. Rev. E **80**, 026111 (2009).

[19] C. Zhou, A.E. Motter, and J. Kurths, Phys. Rev. Lett. **96**, 034101 (2006).

[20] T. Nishikawa and A.E. Motter, Phys. Rev. E **73**, 065106(R) (2006).

[21] T. Nishikawa and A.E. Motter, PNAS (in press); e-print arXiv:0909.2874v1 (2009).

[22] S.M. Park and B.J. Kim, Phys. Rev E **74**, 026114 (2006).

[23] T. Hogg and B.A. Huberman, IEEE Trans. on Sys., Man, and Cybernetics **21**, 1325 (1991).

[24] M.G. Earl and S.H. Strogatz, Phys. Rev. E **67**, 036204 (2003).

[25] S.F. Edwards and D.R. Wilkinson, Proc. R. Soc. London, Ser A **381**, 17 (1982).

[26] R. Frisch and H. Holme, Econometrica **3**, 225 (1935).

[27] N.D. Hayes, J. London Math. Soc. **s1-25**, 226 (1950).

[28] The time discretization of Eq. (5), and in general, that of Eq. (3) naturally has its own effects on the stability through the numerical scheme. Choosing $\Delta t << \tau$ and $\Delta t << 1/\lambda_{\max}$ will yield only small corrections to the behavior of the underlying continuous-time system [36].

[29] Note that this condition coincides with the convergence condition of the deterministic consensus problem [5, 6].

[30] M. Fiedler, Czech. Math. J. **23**, 298 (1973).

[31] W.N. Anderson and T.D. Morley, Lin. Multilin. Algebra **18**, 141 (1985).

[32] A.-L. Barabási and R. Albert, Science **286**, 509 (1999).

[33] R. Albert and A.-L. Barabási, Rev. Mod. Phys. **74**, 47 (2002).

[34] S.N. Dorogovtsev and J.F.F. Mendes, Adv. in Phys. **51**, 1079 (2002).

[35] M. Boguña, and R. Pastor-Satorras, and A. Vespignani, Eur. Phys. J. B **38**, 205 (2004).

[36] D. Hunt, G. Korniss, B.K. Szymanski, to be published.

[37] I.e., the couplings become independent binary random variables such that $C_{ij}(t)=A_{ij}$ with probability p and $C_{ij}(t)=0$ with probability $(1-p)$ at each time step t.