13. Polyhedral Convex Cones

*Mechanics of Manipulation*

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Chapter 1 Manipulation 1
  1.1 Case 1: Manipulation by a human 1
  1.2 Case 2: An automated assembly system 3
  1.3 Issues in manipulation 5
  1.4 A taxonomy of manipulation techniques 7
  1.5 Bibliographic notes 8
    Exercises 8

Chapter 2 Kinematics 11
  2.1 Preliminaries 11
  2.2 Planar kinematics 15
  2.3 Spherical kinematics 20
  2.4 Spatial kinematics 22
  2.5 Kinematic constraint 25
  2.6 Kinematic mechanisms 34
  2.7 Bibliographic notes 36
    Exercises 37

Chapter 3 Kinematic Representation 41
  3.1 Representation of spatial rotations 41
  3.2 Representation of spatial displacements 58
  3.3 Kinematic constraints 68
  3.4 Bibliographic notes 72
    Exercises 72

Chapter 4 Kinematic Manipulation 77
  4.1 Path planning 77
  4.2 Path planning for nonholonomic systems 84
  4.3 Kinematic models of contact 86
  4.4 Bibliographic notes 88
    Exercises 88

Chapter 5 Rigid Body Statics 93
  5.1 Forces acting on rigid bodies 93
  5.2 Polyhedral convex cones 99
  5.3 Contact wrenches and wrench cones 102
  5.4 Cones in velocity twist space 104
  5.5 The oriented plane 105
  5.6 Instantaneous centers and Reuleaux’s method 109
  5.7 Line of force; moment labeling 110
  5.8 Force dual 112
  5.9 Summary 117
  5.10 Bibliographic notes 117
    Exercises 118

Chapter 6 Friction 121
  6.1 Coulomb’s Law 121
  6.2 Single degree-of-freedom problems 123
  6.3 Planar single contact problems 126
  6.4 Graphical representation of friction cones 127
  6.5 Static equilibrium problems 128
  6.6 Planar sliding 130
  6.7 Bibliographic notes 139
    Exercises 139

Chapter 7 Quasistatic Manipulation 143
  7.1 Grasping and fixturing 143
  7.2 Pushing 147
  7.3 Stable pushing 153
  7.4 Parts orienting 162
  7.5 Assembly 168
  7.6 Bibliographic notes 173
    Exercises 175

Chapter 8 Dynamics 181
  8.1 Newton’s Laws 181
  8.2 A particle in three dimensions 181
  8.3 Moment of force; moment of momentum 183
  8.4 Dynamics of a system of particles 184
  8.5 Rigid body dynamics 186
  8.6 The angular inertia matrix 189
  8.7 Motion of a freely rotating body 195
  8.8 Planar single contact problems 197
  8.9 Graphical methods for the plane 203
  8.10 Planar multiple-contact problems 205
  8.11 Bibliographic notes 207
    Exercises 208

Chapter 9 Impact 211
  9.1 A particle 211
  9.2 Rigid body impact 217
  9.3 Bibliographic notes 223
    Exercises 223

Chapter 10 Dynamic Manipulation 225
  10.1 Quasidynamic manipulation 225
  10.2 Brief dynamic manipulation 229
  10.3 Continuously dynamic manipulation 230
  10.4 Bibliographic notes 232
    Exercises 235

Appendix A Infinity 237
Outline.

1. Positive linear span
2. Types of cones
3. Edge and face representation
4. Supplementary cones; polar
5. Representing frictionless contact
6. Cones in wrench space
   - force closure
7. Cones in velocity twist space
Positive linear span

For now, use $n$-dimensional vector space $\mathbb{R}^n$. Later, wrench space and velocity twist space.

Let $\mathbf{v}$ be any non-zero vector in $\mathbb{R}^n$. Then the set of vectors

$$\{k \mathbf{v} \mid k \geq 0\}$$

(1)

describes a ray.

Let $\mathbf{v}_1, \mathbf{v}_2$ be non-zero and non-parallel. Then the set of positively scaled sums

$$\{k_1 \mathbf{v}_1 + k_2 \mathbf{v}_2 \mid k_1, k_2 \geq 0\}$$

(2)

is a planar cone—sector of a plane.

Generalize by defining the positive linear span of a set of vectors $\{\mathbf{v}_i\}$:

$$\text{pos}(\{\mathbf{v}_i\}) = \left\{ \sum k_i \mathbf{v}_i \mid k_i \geq 0 \right\}$$

(3)

(The positive linear span of the empty set is the origin.)
Relatives of positive linear span

The linear span

$$\text{lin}(\{v_i\}) = \{ \sum k_i v_i \mid k_i \in \mathbb{R} \}$$

(4)

The convex hull

$$\text{conv}(\{v_i\}) = \{ \sum k_i v_i \mid k_i \geq 0, \sum k_i = 1 \}$$

(5)
Varieties of cones in three space

1 edge
a. ray

2 edges
b. line c. planar cone

3 edges
d. solid cone e. half plane f. plane

4 edges
g. wedge h. half space i. whole space
Spanning all of $\mathbb{R}^n$

Theorem: A set of vectors $\{\mathbf{v}_i\}$ positively spans the entire space $\mathbb{R}^n$ if and only if the origin is in the interior of the convex hull:

$$\text{pos}(\{\mathbf{v}_i\}) = \mathbb{R}^n \iff 0 \in \text{int}(\text{conv}(\{\mathbf{v}_i\}))$$

(6)

Theorem: It takes at least $n + 1$ vectors to positively span $\mathbb{R}^n$. 

Representing cones

Two ways to represent cones: edge representation and face representation.

Edge representation uses positive linear span. Given a set of edges \( \{e_i\} \), the cone is given by \( \text{pos}(\{e_i\}) \).
Face representation of cones

First represent **planar half-space** by inward pointing normal vector $\mathbf{n}$.

$$\text{half}(\mathbf{n}) = \{ \mathbf{v} \mid \mathbf{n} \cdot \mathbf{v} \geq 0 \}$$  \hspace{1cm} (7)

(Here we use dot product, but when working with twists and wrenches we will use reciprocal product.)

Consider a cone with face normals $\{\mathbf{n}_i\}$. Then the cone is the intersection of the half-spaces:

$$\cap \{\text{half}(\mathbf{n}_i)\}$$  \hspace{1cm} (8)
Supplementary cone; polar

Supplementary cone $\text{supp}(V)$ (also known as polar) comprises the vectors that make non-negative dot products with vectors in $V$:

$$\{ u \in \mathbb{R}^n \mid u \cdot v \geq 0 \ \forall v \in V \}$$  (9)

The supplementary cone’s edges are the original cone’s face normals, and vice versa. So if

$$V = \text{pos}(\{e_i\}) = \cap \{\text{half}(n_i)\}$$  (10)

then

$$\text{supp}(V) = \text{pos}(\{n_i\}) = \cap \{\text{half}(e_i)\}$$  (11)
Frictionless contact

Characterize contact by *set of possible wrenches.*

Assume uniquely determined contact normal.

Assume frictionless contact can give arbitrary magnitude force along inward-pointing normal.

Then a frictionless contact gives a ray in wrench space, $\text{pos}(w)$, where $w = (c, c_0)$ is the contact screw.
Two contacts

Given two frictionless contacts $w_1$ and $w_2$, total wrench is the sum of possible positive scalings of $w_1$ and $w_2$:

$$k_1 w_1 + k_2 w_2; k_1, k_2 \geq 0$$

(12)

i.e. the positive linear span $\text{pos}([w_1, w_2])$.

Generalizing:

Theorem: If a set of frictionless contacts on a rigid body is described by the contact normals $w_i = (c_i, c_0i)$ then the set of all possible wrenches is given by the positive linear span $\text{pos}([w_i])$. 
**Force closure**

Definition: **Force closure** means that the set of possible wrenches exhausts all of wrench space.

It follows from theorem ? that a frictionless force closure requires at least 7 contacts. Or, since planar wrench space is only three-dimensional, frictionless force closure in the plane requires at least 4 contacts.
Example wrench cone

Construct unit magnitude force at each contact.
Write screw coords of wrenches.
Take positive linear span.
Exhausts wrench space?
Cones in velocity twist space

Cannot use finite displacement twists. They are not vectors.
Velocity twists are vectors!
Let \( \{ w_i \} \) be a set of contact normals.
Let \( W = \text{pos}(\{ w_i \}) \) be set of possible wrenches.
First order analysis: velocity twists \( T \) must be *reciprocal or repelling*
to contact wrenches: \( T = \text{supp}(W) \).