2. Kinematic foundations

*Mechanics of Manipulation*

Matt Mason
matt.mason@cs.cmu.edu

http://www.cs.cmu.edu/~mason

Carnegie Mellon
Kinematic foundations.

We will focus on rigid motions in

the Euclidean plane \((\mathbb{E}^2)\)

Euclidean three space \((\mathbb{E}^3)\)

the sphere \((\mathbb{S}^2)\)

Why the sphere? Rigid motions of the sphere correspond to rotations about a given point in \(\mathbb{E}^3\).
Kinematics foundations: some definitions

First, some general definitions. Let \( X \) be the *ambient space*, either \( \mathbb{E}^2 \), \( \mathbb{E}^3 \), or \( S^2 \).

- A *system* is a set of points in the space \( X \).
- A *configuration* of a system is the location of every point in the system.
- *Configuration space* is a metric space comprising all configurations of a given system. (What kind of space is configuration space? Devise a metric.) (Note: Every metric for cspace is sort of defective.)
- The *degrees of freedom* of a system is the dimension of the configuration space. (A less precise but roughly equivalent definition: the minimum number of real numbers required to specify a configuration.)
**Kinematics foundations: systems, DOFs**

<table>
<thead>
<tr>
<th>System</th>
<th>Configuration</th>
<th>DOFs</th>
</tr>
</thead>
<tbody>
<tr>
<td>point in plane</td>
<td>$x, y$</td>
<td>2</td>
</tr>
<tr>
<td>point in space</td>
<td>$x, y, z$</td>
<td>3</td>
</tr>
<tr>
<td>rigid body in plane</td>
<td>$x, y, \theta$</td>
<td>3</td>
</tr>
<tr>
<td>rigid body in space</td>
<td>$x, y, z, \phi, \theta, \psi$</td>
<td>6</td>
</tr>
</tbody>
</table>
Kinematics foundations: rigid bodies, displacement

Definitions:

A *displacement* is a change of configuration that does not change the distance between any pair of points, nor does it change the handedness of the system.

A *rigid body* is a system that is capable of displacements only.

Transformations, rigid and otherwise.
Kinematics foundations: moving and fixed planes

We will consider displacements to apply to every point in the ambient space. E.g., displacements are described as motion of moving plane relative to fixed plane.
Kinematics foundations: rotations and translations

A rotation is a displacement that leaves at least one point fixed. A translation is a displacement for which all points move equal distances along parallel lines.

- Rotation about \( O \)
- Rotation about a point on the body
- Rotation about a point not on the body
Kinematics foundations: digression for group theory

A group is a set of elements $X$ and a binary operator $\circ$ satisfying the following properties:

**Closure:** for all $x$ and $y$ in $X$, $x \circ y$ is in $X$.

**Associativity:** for all $x$, $y$, and $z$ in $X$, $(x \circ y) \circ z$ is equal to $x \circ (y \circ z)$.

**Identity:** there is some element, called 1, such that for all $x$ in $X$ $x \circ 1 = 1 \circ x = x$.

**Inverses:** for all $x$ in $X$, there is some element called $x^{-1}$ such that $x \circ x^{-1} = x^{-1} \circ x = 1$.

(Did I remember them all?)

Some groups are commutative (Abelian) and some are not. The integers with addition are a commutative group. Nonsingular $k$ by $k$ matrices with matrix multiplication are a noncommutative group.
Kinematics foundations: Displacements as a group

Every displacement $D$ can be described as an operator on the ambient space $X$, mapping every point $x$ to some new point $D(x) = x'$. The product of two displacements is the composition of the corresponding operators, i.e. $(D_2 \circ D_1)(\cdot) = D_2(D_1(\cdot))$. The inverse of a displacement is just the operator that maps every point back to its original position. The identity is the null displacement, which maps every point to itself.

In other words:

The displacements, with functional composition, form a group.
Kinematics foundations: $\text{SE}(2)$, $\text{SE}(3)$, and $\text{SO}(3)$

These groups of displacements have names:

$\text{SE}(2)$: The special Euclidean group on the plane.

$\text{SE}(3)$: The special Euclidean group on $\mathbb{E}^3$.

$\text{SO}(3)$: The special orthogonal group.

Whence the names?

**Special**: they preserve handedness.

**Orthogonal**: referring to the connection with orthogonal matrices, which will be covered later.
Kinematics foundations: do displacements commute?

Does $\text{SO}(3)$ commute? **NO!** No, no, no. (If you have found a commutative way of representing spatial rotations, you are confused.)
Kinematics foundations: do displacements commute?

Does $\text{SE}(3)$ commute?
Does $\text{SE}(2)$ commute?
Does $\text{SO}(2)$ commute?
Time for a digression . . .

Next we look at $\mathbf{SE}(2)$, $\mathbf{SO}(3)$, and $\mathbf{SE}(3)$.

First, it helps if we contemplate the infinite . . .
The projective plane.

The basic idea:

Start with the Euclidean plane $\mathbb{E}^2$.

Add some points, the ideal points or the points at infinity.

Call the new structure the projective plane $\mathbb{P}^2$.

You can do it formally by defining an ideal point for each set of parallel lines, but we will employ a more concrete method using homogeneous coordinates.
Homogeneous coordinates.

Let the Cartesian coordinates of some point in $\mathbb{E}^2$ be

$$(\eta, \nu)$$

Then we will say that

$$(x, y, w) \triangleq (w\eta, w\nu, w)$$

are the homogeneous coordinates of the point, provided

$$w \neq 0$$

To go from homogeneous to Cartesian:

$$\begin{pmatrix} x \\ y \\ w \end{pmatrix} \mapsto \begin{pmatrix} x/w \\ y/w \end{pmatrix}, \quad w \neq 0 \quad (1)$$
Point in $\mathbb{E}^2$ versus line through origin of $\mathbb{E}^3$

Scaling the homogeneous coordinates does **not** change the point!

\[
\begin{pmatrix} ax \\ ay \\ aw \end{pmatrix} \mapsto \begin{pmatrix} ax/aw \\ ay/aw \end{pmatrix} = \begin{pmatrix} x/w \\ y/w \end{pmatrix}, \ a, w \neq 0 \tag{2}
\]

So, homogeneous coordinates represent a point in $\mathbb{E}^2$ by a line through the origin of $\mathbb{E}^3$.

\[
\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \left\{ \begin{pmatrix} wx \\ wy \\ w \end{pmatrix} \middle| \ w \neq 0 \right\}
\]
Central projection

The Euclidean plane can be embedded as the $w = 1$ plane.
We can also embed a sphere of points satisfying $x^2 + y^2 + w^2 = 1$.
A line through the origin of $\mathbb{E}^3$

intersects the sphere in antipodal points

intersects the $w = 1$ plane at the appropriate point $(x/w, y/w)$.

These constructions are central projection, either to the sphere or to the plane.
Ideal points

The original idea: extend $\mathbb{E}^2$ by adding some ideal points.

Euclidean point: line through origin of $\mathbb{E}^3$ intersecting $w = 1$ plane.

Ideal point: line through origin of $\mathbb{E}^3$ parallel to $w = 1$ plane.

With Cartesian coords, no place to put ideal points. With homogeneous coordinates, there’s a big gaping hole!
The projective plane

So ... define the projective plane $\mathbb{P}^2$ to be the set of lines through the origin of $\mathbb{E}^3$.

A line in $\mathbb{E}^2$ is represented by plane through origin of $\mathbb{E}^3$.

The ideal points form a line! The line at infinity. The equator of the embedded sphere.

"Parallel lines" intersect at infinity.

Duality. Two points determine a line. Two lines determine a point. Every axiom of the projective plane has a dual axiom by switching "line" and "point".

Noneuclidean geometry!!!