24. Rigid Body Dynamics

Mechanics of Manipulation

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First a quiz

Outline.

Newtonian mechanics of a single particle;

of a system of several particles;

of a rigid body.

Newton's laws

1. Every body continues at rest, or in uniform motion in a straight line, unless forces act upon it.
2. The rate of change of momentum is proportional to the applied force.
3. The forces acting between two bodies are equal and opposite.

Define **momentum** to be mass times velocity.

Consider a particle . . .

. . . of mass m,

with position represented by a vector $\mathbf{x}$,

total applied force $\mathbf{F}$,

momentum

$$\mathbf{p} = m\mathbf{v} = m\frac{d\mathbf{x}}{dt}$$

so Newton's second law can be written

$$m\frac{d^2\mathbf{x}}{dt^2} = \mathbf{F}$$

Impulse, kinetic energy

Integrating Newton's second law:

$$\mathbf{p}_2 - \mathbf{p}_1 = \int_{t_1}^{t_2} \mathbf{F}\, dt$$

stating that the change in momentum is equal to the *impulse*.

We can also define *kinetic energy* T

$$T = \frac{m}{2} |\mathbf{v}|^2$$

Power

Differentiating kinetic energy yields

$$\begin{aligned}\frac{dT}{dt} &= \frac{m}{2}\frac{d}{dt}(\mathbf{v}\cdot\mathbf{v}) \\ &= \frac{m}{2}\left(\frac{d\mathbf{v}}{dt}\cdot\mathbf{v}+\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}\right) \\ &= m\frac{d\mathbf{v}}{dt}\cdot\mathbf{v} \\ &= \mathbf{F}\cdot\mathbf{v}\end{aligned}$$

stating that the time rate of change of kinetic energy is *power*.

Work

Integrating the power over a time interval,

$$T_2 - T_1 = \int_{t_1}^{t_2} \mathbf{F} \cdot \mathbf{v} \, dt$$

or

$$T_2 - T_1 = \int_{\mathbf{x}_1}^{\mathbf{x}_2} \mathbf{F} \cdot \mathbf{dx}$$

stating that the change in kinetic energy is *work*.

Moment of force; moment of momentum

Recall definition of moment of force about a point $\mathbf{x}$:

$$\mathbf{n} = \mathbf{x} \times \mathbf{f}$$

and about a line l through origin with direction $\hat{\mathbf{l}}$

$$n_l = \hat{\mathbf{l}} \cdot \mathbf{n}$$

Similarly, suppose a particle at $\mathbf{x}$ has momentum $\mathbf{p}$.

- Define **moment of momentum** about the origin

$$\mathbf{L} = \mathbf{x} \times \mathbf{p}$$

- and about the line l

$$L_l = \hat{\mathbf{l}} \cdot \mathbf{L}$$

Rate of change of moment of momentum

Differentiating the moment of momentum:

$$\begin{aligned}\frac{d\mathbf{L}}{dt} &= \frac{d}{dt}(\mathbf{x} \times \mathbf{p}) \\ &= \frac{d}{dt}(\mathbf{x} \times m\mathbf{v}) \\ &= m\left(\frac{d\mathbf{x}}{dt} \times \mathbf{v} + \mathbf{x} \times \frac{d\mathbf{v}}{dt}\right) \\ &= \mathbf{x} \times m\frac{d\mathbf{v}}{dt} \\ &= \mathbf{x} \times \mathbf{F} \\ &= \mathbf{N}\end{aligned}$$

which is essentially a restatement of Newton's second law, but using moments of force and momentum.

So, for a particle ...

Using either $\mathbf{F} = d\mathbf{p}/dt$ or $\mathbf{N} = d\mathbf{L}/dt$, we have three second order differential equations.

If $\mathbf{F}$ or $\mathbf{N}$ is uniquely determined by the state $(\mathbf{x}, \mathbf{v})$, then there is a unique solution giving $\mathbf{x}(t)$ and $\mathbf{v}(t)$ for any given initial conditions $\mathbf{x}(0) = \mathbf{x}_0$, $\mathbf{v}(0) = \mathbf{v}_0$.

For a bunch of particles

For the kth particle

- Let m_k be the mass,
- let $\mathbf{x}_k$ be the position vector,
- and let $\mathbf{p}_k$ be the momentum.
- Let the force be composed of internal force (from interactions with other particles in the system) and external forces $\mathbf{F}_k = \mathbf{F}_k^i + \mathbf{F}_k^e$

Momentum and force

We define the momentum of the system to be

$$\mathbf{P} = \sum \mathbf{p}_k$$

and the total force on the system to be

$$\mathbf{F} = \sum \mathbf{F}_k^e$$

(The sum of all internal forces is zero, by Newton's third law.)

Newton's 2nd law for system of particles

Newton's 2nd law for kth particle:

$$\frac{d\mathbf{p}_k}{dt} = \mathbf{F}_k^e + \mathbf{F}_k^i$$

Summing:

$$\sum \frac{d\mathbf{p}_k}{dt} = \sum \left(\mathbf{F}_k^e + \mathbf{F}_k^i\right)$$

Hence

$$\frac{d\mathbf{P}}{dt} = \mathbf{F}$$

Newton's second law extends to the system of particles.

Center of mass

Define total mass:

$$M = \sum m_k$$

and the center of mass,

$$\mathbf{X} = \frac{1}{M} \sum m_k \mathbf{x}_k$$

Then

$$\mathbf{P} = M \frac{d\mathbf{X}}{dt}$$

and

$$\mathbf{F} = M \frac{d^2\mathbf{X}}{dt^2}$$

which means that the center of mass behaves just like a single particle.

Moments for systems of particles

Define $\mathbf{L}_k$ to be the angular momentum of the kth point,

Define the total angular momentum to be the sum,

$$\mathbf{L} = \sum \mathbf{L}_k$$

Define the total torque,

$$\mathbf{N} = \sum \mathbf{x}_k \times \mathbf{F}_k^e$$

Rate of change of moment of momentum

Now for the kth particle

$$\frac{d\mathbf{L}_k}{dt} = \mathbf{x}_k \times \mathbf{F}_k^e + \mathbf{x}_k \times \mathbf{F}_k^i$$

Summing over all the particles,

$$\frac{d\mathbf{L}}{dt} = \mathbf{N} + \sum \mathbf{x}_k \times \mathbf{F}_k^i$$

By Newton's third law the sum of the internal moments is zero, so that the second term vanishes:

$$\frac{d\mathbf{L}}{dt} = \mathbf{N}$$

which is grand, but six equations is not enough to determine the motion of several particles.

Rigid body dynamics

A rigid body is a bunch of particles, but with all distances fixed. Six degrees of freedom. Wouldn't it be keen if the six equations

$$\mathbf{F} = d\mathbf{P}/dt$$

$$\mathbf{N} = d\mathbf{L}/dt$$

were enough?

Angular inertia, part one

For a rigid body, velocity of kth particle is

$$\mathbf{v} = \mathbf{v}_0 + \omega \times \mathbf{x}$$

Substituting into moment of momentum

$$\mathbf{L}_k = m_k \mathbf{x}_k \times (\mathbf{v}_0 + \omega \times \mathbf{x}_k)$$

Summing to obtain the total angular momentum,

$$\begin{aligned}\mathbf{L} &= \sum m_k \mathbf{x}_k \times \mathbf{v}_0 + \sum m_k \mathbf{x}_k \times (\omega \times \mathbf{x}_k) \\ &= M\mathbf{X} \times \mathbf{v}_0 + \sum m_k \mathbf{x}_k \times (\omega \times \mathbf{x}_k)\end{aligned}$$

Place origin at center of mass to eliminate first term on right

$$\mathbf{L} = \sum m_k \mathbf{x}_k \times (\omega \times \mathbf{x}_k)$$

Angular inertia, part two

How can we get that pesky ω out of the sum?

$$\mathbf{L} = \sum m_k \mathbf{x}_k \times (\omega \times \mathbf{x}_k)$$

Applying the identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$,

$$\mathbf{L} = \sum m_k \left[(\mathbf{x}_k \cdot \mathbf{x}_k)\omega - \mathbf{x}_k(\mathbf{x}_k \cdot \omega) \right]$$

Represent each vector as a column matrix, and substitute $\mathbf{x}_k^t \omega$ for $\mathbf{x}_k \cdot \omega$:

$$\mathbf{L} = \left(\sum m_k \left(|\mathbf{x}_k|^2 I_3 - \mathbf{x}_k \mathbf{x}_k^t \right) \right) \omega$$

where I_3 is the three-by-three identity matrix.

Angular inertia part three

Define the *angular inertia matrix* I:

$$I = \sum m_k \left(|\mathbf{x}_k|^2 I_3 - \mathbf{x}_k \mathbf{x}_k^t \right)$$

Substituting above,

$$\mathbf{L} = I\omega$$