25. Tumbling rigid bodies

Mechanics of Manipulation

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Outline.

- The angular inertia tensor;
- Euler's equations;
- Poinsot's construction for a tumbling rigid body.

Last time . . .

. . . we defined the *angular inertia matrix* I:

$$I = \sum m_k \left(|\mathbf{x}_k|^2 I_3 - \mathbf{x}_k \mathbf{x}_k^t \right)$$

so that moment of momentum is

$$\mathbf{L} = I\omega$$

where Newton's second law gives

$$\mathbf{N} = \frac{d\mathbf{L}}{dt}$$

Differentiating $\mathbf{L} = I\omega$

I is constant in the *body* frame, not in an inertial frame.

$$\mathbf{N} = \frac{d(I\omega)}{dt} \tag{1}$$

$$= I\frac{d\omega}{dt} + \frac{dI}{dt}\omega \tag{2}$$

$$= I\frac{d\omega}{dt} + \omega \times (I\omega) \tag{3}$$

Zero torque implies constant angular momentum.

Zero torque does *not* imply constant angular velocity.

What *can* you say about how angular velocity changes? First we need to look closer at the angular inertia tensor.

Coordinate frame issue in differentiation

Did we differentiate $\mathbf{L}$ in an inertial frame? Origin coinciding with possibly accelerating center of mass? Yes! We used inertial frame instantaneously coincident with body frame.

That's inconvenient: If we solved for $\frac{d\omega}{dt}$, it would be expressed in a different frame for every value of t.

Convert Euler's equations to moving body frame from coincident fixed frame. $\mathbf{N}$, I, ω unchanged. New angular acceleration is

$$\frac{d\omega}{dt} + \omega \times \omega$$

I.e., Euler's equations work in the moving body frame. Yay.

The inertia tensor

Let body be continuous with density ρ.

$$I = \int \rho \left(|\mathbf{x}|^2 I_3 - \mathbf{x}\mathbf{x}^t \right) \, dV \tag{4}$$

In components:

$$I = \int \rho \begin{pmatrix} x_2^2 + x_3^2 & -x_1 x_2 & -x_1 x_3 \\ -x_1 x_2 & x_1^2 + x_3^2 & -x_2 x_3 \\ -x_1 x_3 & -x_2 x_3 & x_1^2 + x_2^2 \end{pmatrix} dV \tag{5}$$

Moments of inertia; products of inertia

Diagonal elements: moments of inertia w.r.t. the coordinate axes:

$$I_{11} = \int \rho(x_2^2 + x_3^2)\, dV \tag{6}$$

etc. (7)

Off-Diagonal elements: the *products of inertia*:

$$I_{12} = I_{21} = -\int \rho x_1 x_2\, dV \tag{8}$$

etc. (9)

We could try to understand them, or we could get rid of them ...

Principal axes; principal moments of inertia

Inertia matrix is symmetric—diagonal in the right frame. Define A:

$$ {}^{A}I = \begin{pmatrix} {}^{A}I_{11} & 0 & 0 \\ 0 & {}^{A}I_{22} & 0 \\ 0 & 0 & {}^{A}I_{33} \end{pmatrix} \tag{10} $$

I in A-coordinates can be obtained by:

$$ {}^{A}I = AIA^{T} \tag{11} $$

where matrix A transforms to A-coordinates.

principal axes: coordinate axes of A—eigenvectors of I.

principal moments: diagonal elements of ${}^{A}I$—eigenvalues of I.

Distinct eigenvalues implies uniquely determined principal axes.

Scalar angular inertia. Radius of gyration.

Consider moment of momentum $\mathbf{L} = I\omega$. When are $\mathbf{L}$ and ω parallel?

Consider rotation about some fixed axis in direction $\hat{\mathbf{n}}$. Scalar angular inertia I_n is

$$I_n = \hat{\mathbf{n}}^t I \hat{\mathbf{n}} \tag{12}$$

radius of gyration k_n with respect to the axis $\hat{\mathbf{n}}$:

$$I_n = Mk_n^2 \tag{13}$$

The radius of gyration represents the distance of a point mass that would give the same angular inertia.

Inertia ellipsoid

Consider the surface described by the equation

$$\mathbf{r}^t I \mathbf{r} = a \tag{14}$$

In principal coordinates, since moments are positive, we get an ellipsoid:

$$I_{xx} r_x^2 + I_{yy} r_y^2 + I_{zz} r_z^2 = a \tag{15}$$

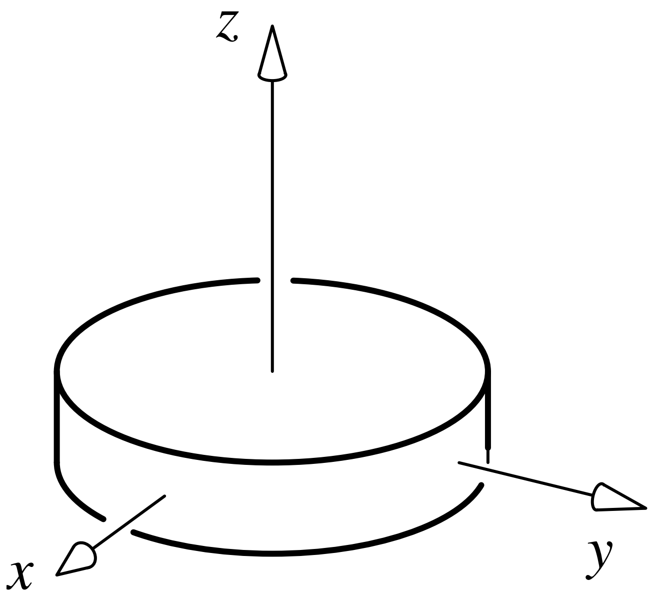
Let $\mathbf{r} = r\hat{\mathbf{n}}$. Then

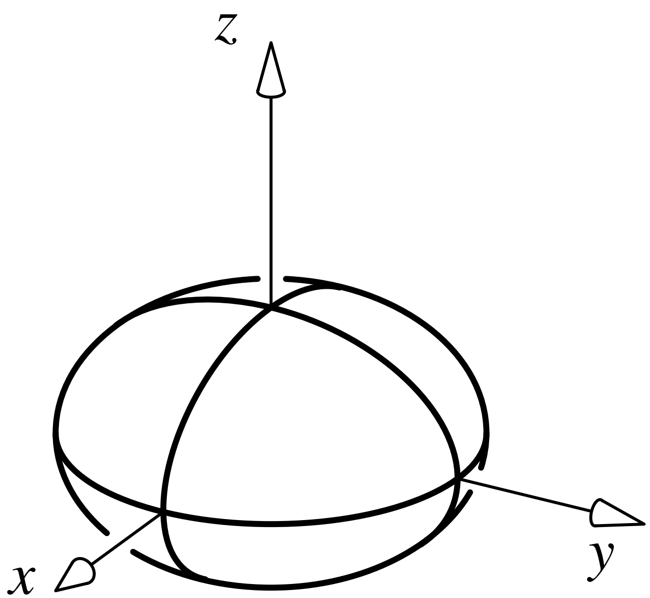
$$I_n = \hat{\mathbf{n}}^t I \hat{\mathbf{n}} = \frac{1}{r^2}\mathbf{r}^t I \mathbf{r} = \frac{a}{r^2} \tag{16}$$

So distance to ellipsoid surface is inverse of radius of gyration.

$$M k_n^2 = \frac{a}{r^2} \tag{17}$$

Cylinder and its inertia ellipsoid





Principal axes by inspection

There are theorems:

- Any plane of symmetry is perpendicular to a principal axis.
- Any line of symmetry is a principal axis.

If you start in the principal frame, you know the products of inertia are zero, so you can get the inertia tensor by just doing three integrals.

Rigid body tumbling

$$N = \begin{pmatrix} I_1\dot{\omega}_1 \\ I_2\dot{\omega}_2 \\ I_3\dot{\omega}_3 \end{pmatrix} + \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \times \begin{pmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{pmatrix} \tag{18}$$

$$= \begin{pmatrix} I_1\dot{\omega}_1 + (I_3\omega_2\omega_3 - I_2\omega_2\omega_3) \\ I_2\dot{\omega}_2 + (I_1\omega_3\omega_1 - I_3\omega_3\omega_1) \\ I_3\dot{\omega}_3 + (I_2\omega_1\omega_2 - I_1\omega_1\omega_2) \end{pmatrix} \tag{19}$$

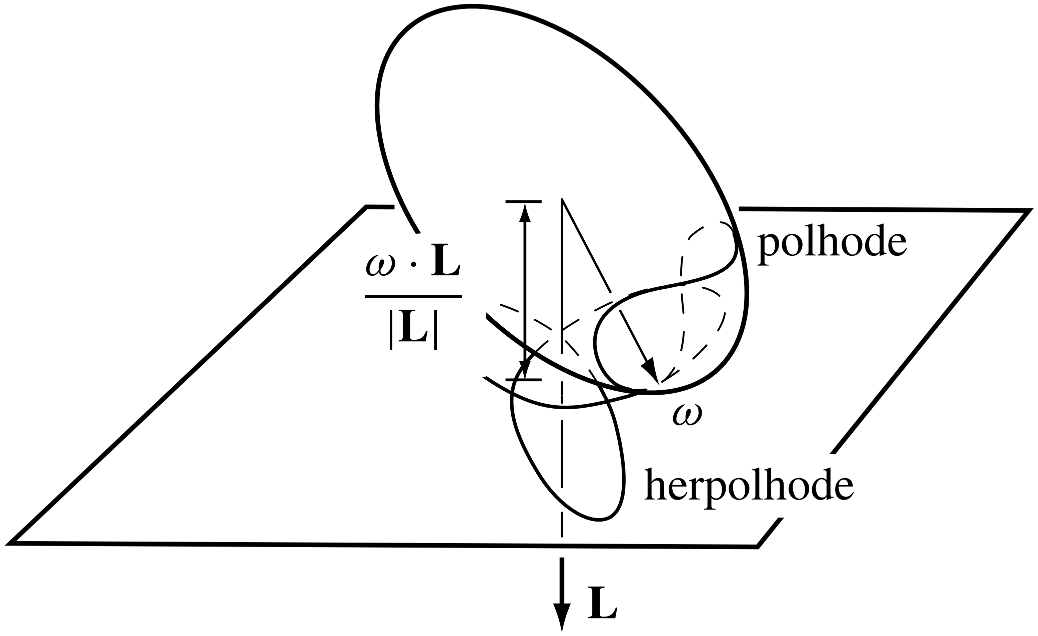
If $N = 0$ we get what looks like Euler's equations:

$$\dot{\omega}_1 = \frac{I_2 - I_3}{I_1}\omega_2\omega_3 \tag{20}$$

$$\dot{\omega}_2 = \frac{I_3 - I_1}{I_2}\omega_3\omega_1 \tag{21}$$

$$\dot{\omega}_3 = \frac{I_1 - I_2}{I_3}\omega_1\omega_2 \tag{22}$$

Poinsot's construction



Rigid body tumbling: inertia ellipsoid rolls without slipping on a plane.

Proof of Poinsot's construction

If N is zero, then kinetic energy T is constant:

$$T = \frac{1}{2}\omega^t I \omega \quad \text{is constant} \tag{23}$$

That is, ω is on the surface of the inertia ellipsoid.

What is the *tangent plane* normal at ω?

$$\nabla \frac{1}{2}\omega^t I \omega = \nabla \frac{1}{2}(\omega_1^2 I_1 + \omega_2^2 I_2 + \omega_3^2 I_3) \tag{24}$$

$$= (I_1\omega_1, I_2\omega_2, I_3\omega_3) = \mathbf{L} \tag{25}$$

The attitude of the tangent plane is constant!

How far from center of mass to tangent plane?

$$\frac{\omega \cdot \mathbf{L}}{|\mathbf{L}|} = \frac{2T}{|\mathbf{L}|} \tag{26}$$

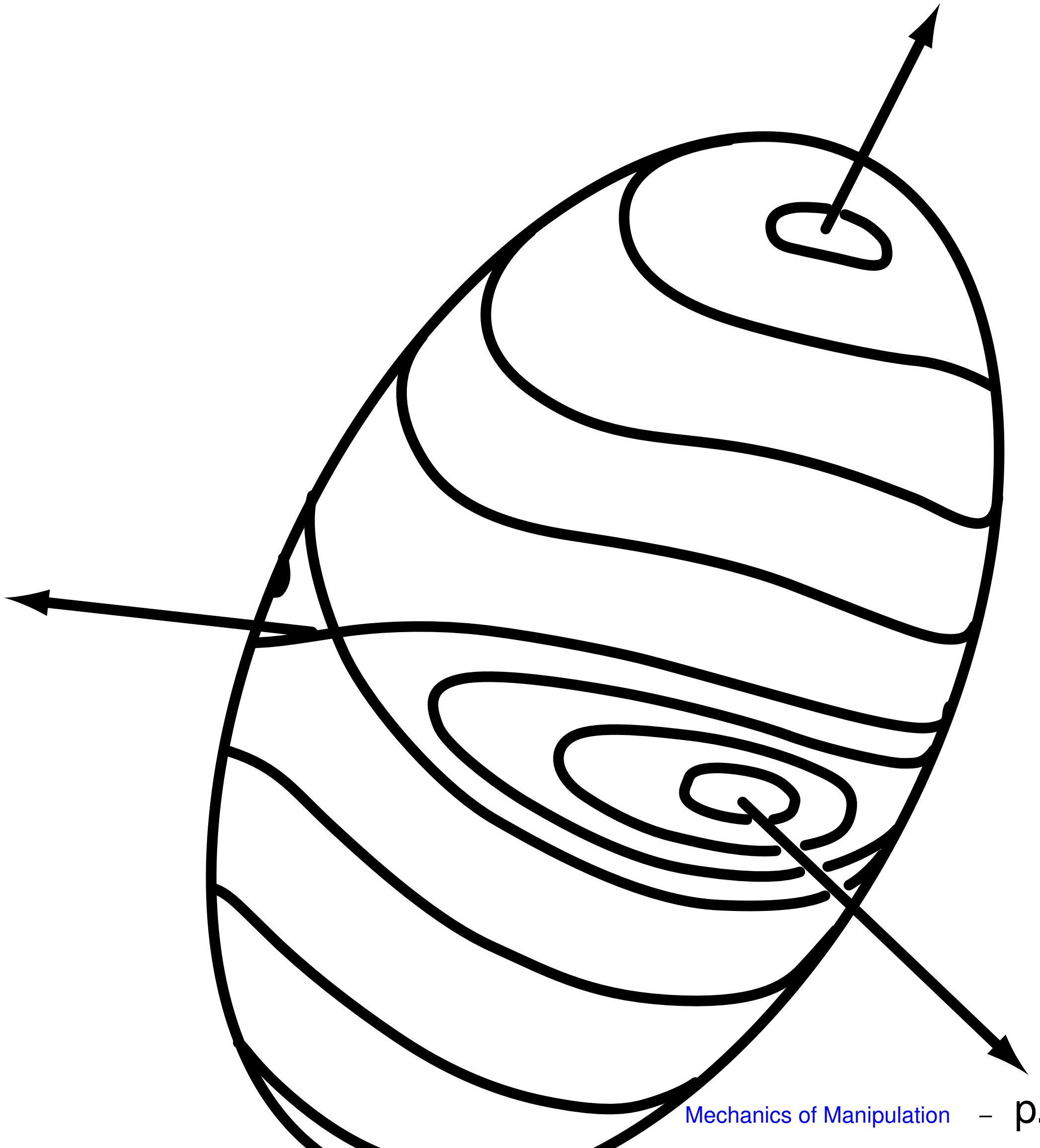
polhodes

Take an ellipsoid, hold the center a fixed distance from an inkpad, and roll it around.

Near the pointy end you get little loops.

Near the center of mass you get little loops.

Near the third principal axis, you get sent away.



Video of tumbling body