27. Impact

Mechanics of Manipulation

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Outline.

- Impulse-momentum equations
- Impact of a particle
  - Plastic and elastic impact
  - Newtonian and Poisson restitution
  - Impulse space
- Impact of a planar rigid body
  - Solution of the sliding rod problem

Frictional impact

When rigid bodies collide:

- Discontinuity in velocity,
- Infinite forces,
- Zero time,
- Even for simple cases there is no agreement on laws,
- For multiple contact indeterminacies abound.

We will adopt Routh's analysis of planar impact, which uses Poisson restitution.

Impulse momentum equations

Recall:

$$\mathbf{F} = \frac{d\mathbf{P}}{d\mathbf{t}}$$

$$\mathbf{N} = \frac{d\mathbf{L}}{d\mathbf{t}}$$

Integrating:

$$\Delta\mathbf{P} = \int_{t_0}^{t_1} \mathbf{F}\, dt$$

$$\Delta\mathbf{L} = \int_{t_0}^{t_1} \mathbf{N}\, dt$$

where $\Delta\mathbf{P} = \mathbf{P}_1 - \mathbf{P}_0$, $\Delta\mathbf{L} = \mathbf{L}_1 - \mathbf{L}_0$, $\int \mathbf{F}\, dt$ is the *impulse*, and $\int \mathbf{N}\, dt$ is the *impulsive moment*.

Determining impulse. Particle in plane.

We need impulse and impulsive moment as a function of state and physical parameters.

Start with a frictionless particle on plane. Velocity changes discontinuously from $\mathbf{v}_0$ to $\mathbf{v}_1$:

$$\Delta \mathbf{v} = \frac{1}{m} \Delta \mathbf{P} = \frac{1}{m} \mathbf{I}$$

where

$$\Delta \mathbf{v} = \mathbf{v}_1 - \mathbf{v}_0$$

The tangential impulse I_t is zero, so

$$v_{1t} = v_{0t}$$

$$v_{1n} = v_{0n} + \frac{1}{m} I_n$$

Plastic and elastic

How to get the normal impulse I_n? Two special cases: *plastic impact*

$$I_n = -mv_{0n}$$
$$\rightarrow v_{1n} = 0$$

and *elastic impact*

$$I_n = -2mv_{0n}$$
$$\rightarrow v_{1n} = -v_{0n}$$

Newtonian restitution

Newton hypothesized a continuum from plastic to elastic, defined by a *coefficient of restitution*

$$e = -\frac{v_{1n}}{v_{0n}} \tag{1}$$

so that plastic impact corresponds to $e = 0$, and elastic impact corresponds to $e = 1$.

Poisson restitution

A competing definition of the coefficient of restitution was given by Poisson. Divide the collision into two stages.

$v_n < 0$	compression
$v_n = 0$	
$v_n > 0$	restitution

Let I_c be impulse accumulated during compression, and let I_r be impulse accumulated during restitution. Poisson's hypothesis is that the ratio of these two parts is governed by material properties:

$$e = \frac{I_r}{I_c} \tag{2}$$

For simple cases no advantage over Newton, but for frictional rigid bodies Poisson restitution is superior.

Applying Poisson restitution for frictionless parti

Given m, $\mathbf{v}_0$, and e, first find I_c:

$$I_c = mv_{0n}$$

then restitution impulse I_r

$$I_r = emv_{0n}$$

then total impulse I

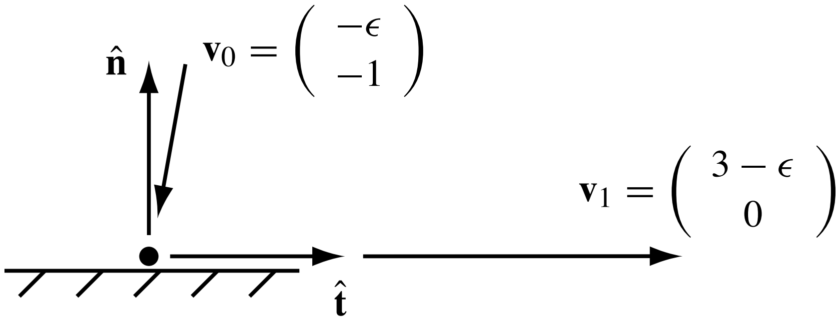
$$I = I_c + I_r = (1 + e)mv_{0n}$$

and resulting normal velocity:

$$v_{1n} = -ev_{0n}$$

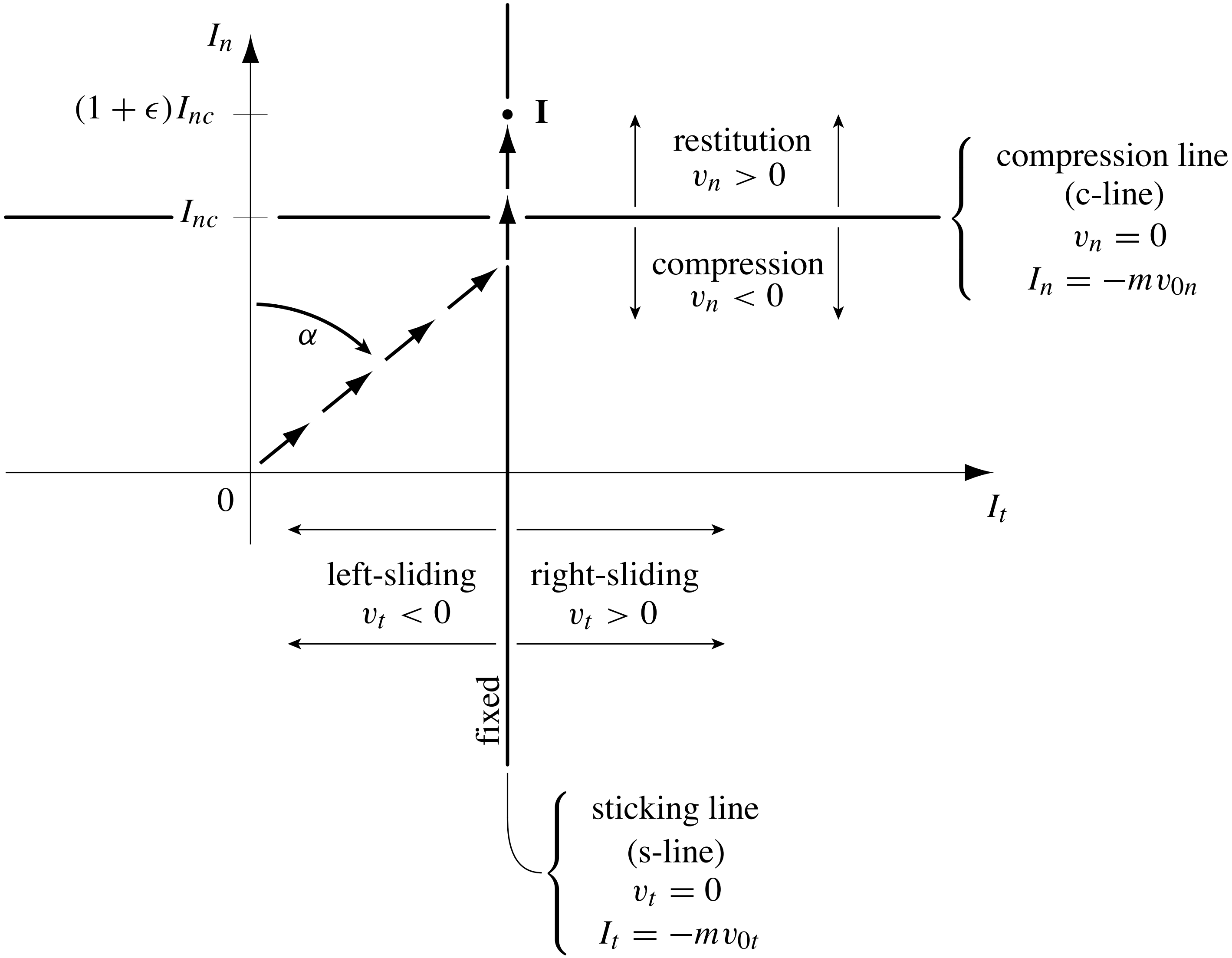
The tangential velocity is unchanged.

Frictional particle



Consider frictional particle with initial leftward velocity. There should be some frictional impulse acting toward the right. Suppose we apply Coulomb's law, by looking at initial contact mode, and multiplying I_n by μ. The result is absurd, and violates conservation of energy!

A better model



The left-drifting particle

$$\mathbf{v}_0 = \begin{pmatrix} -\epsilon \\ -1 \end{pmatrix}$$

$e = 0$

$\mu = 3$

$\alpha = \tan^{-1} 3$

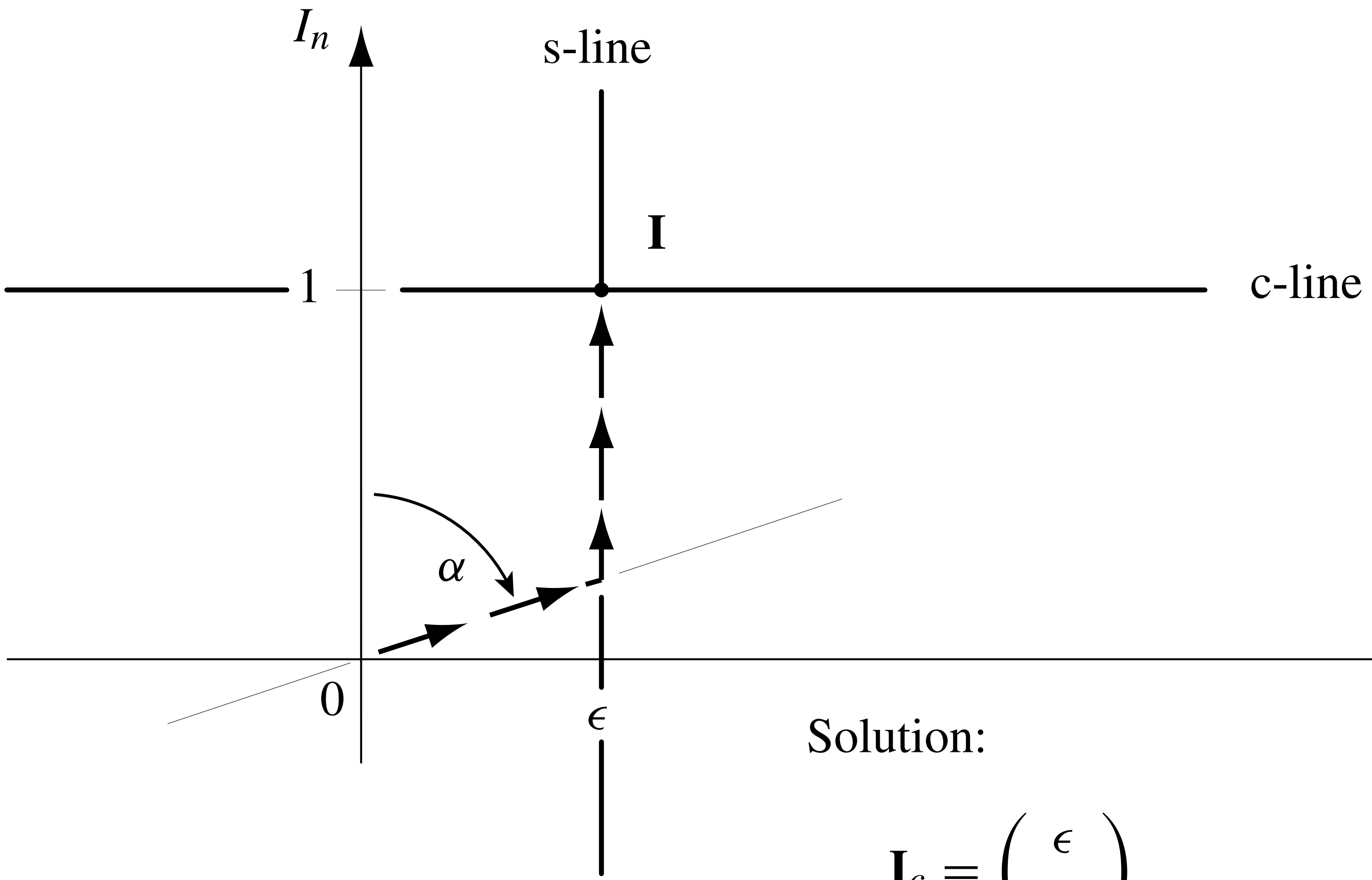
$m = 1$

c-line:

$I_n = -m v_{0n} = 1$

s-line:

$I_t = -m v_{0t} = \epsilon$



Solution:

$$\mathbf{I}_c = \begin{pmatrix} \epsilon \\ 1 \end{pmatrix}$$

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Rigid body. Kinematics.

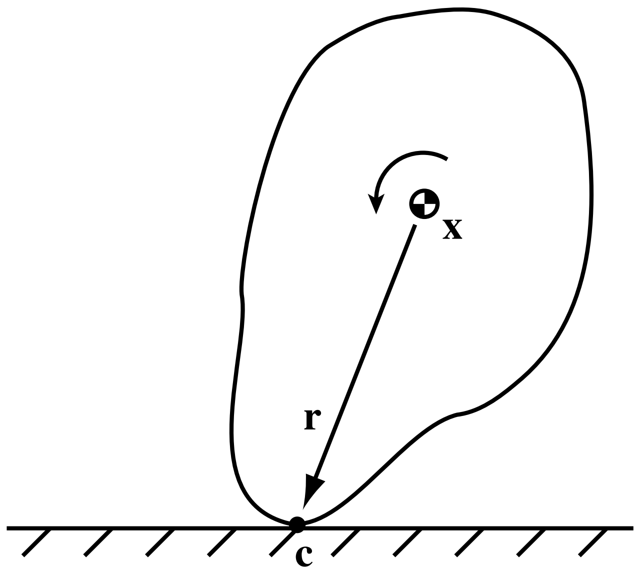
First, we observe some kinematic relations

$$\mathbf{c} = \mathbf{x} + \mathbf{r}$$

$$\dot{\mathbf{c}} = \dot{\mathbf{x}} + \dot{\mathbf{r}}$$

$$= \dot{\mathbf{x}} + \omega \times \mathbf{r}$$

$$\Delta\dot{\mathbf{c}} = \Delta\dot{\mathbf{x}} + \Delta\omega \times \mathbf{r}$$



Impulse momentum laws

Now we can write the impulse-momentum laws:

$$m\Delta\dot{\mathbf{x}} = \mathbf{I} \tag{3}$$

$$\rho^2 m\Delta\omega = \mathbf{r} \times \mathbf{I} \tag{4}$$

Substituting into the expression for $\Delta\dot{\mathbf{c}}$ yields:

$$\Delta\dot{\mathbf{c}} = \frac{1}{m}\mathbf{I} + \frac{1}{\rho^2 m}(\mathbf{r} \times \mathbf{I}) \times \mathbf{r} \tag{5}$$

$$= \frac{1}{m}\mathbf{I} - \frac{1}{\rho^2 m}\mathbf{r} \times (\mathbf{r} \times \mathbf{I}) \tag{6}$$

$$= \frac{1}{\rho^2 m}\left(\rho^2 I_3 - R^2\right)\mathbf{I} \tag{7}$$

where I_3 is the three by three identity matrix, and R is the cross-product matrix.

Compression line and restitution line

Substituting above and expanding, we obtain

$$\begin{pmatrix} \Delta\dot{c}_t \\ \Delta\dot{c}_n \end{pmatrix} = \frac{1}{\rho^2 m} \begin{pmatrix} (\rho^2 + r_n^2) & -r_t r_n \\ -r_t r_n & (\rho^2 + r_t^2) \end{pmatrix} \begin{pmatrix} I_t \\ I_n \end{pmatrix} \tag{8}$$

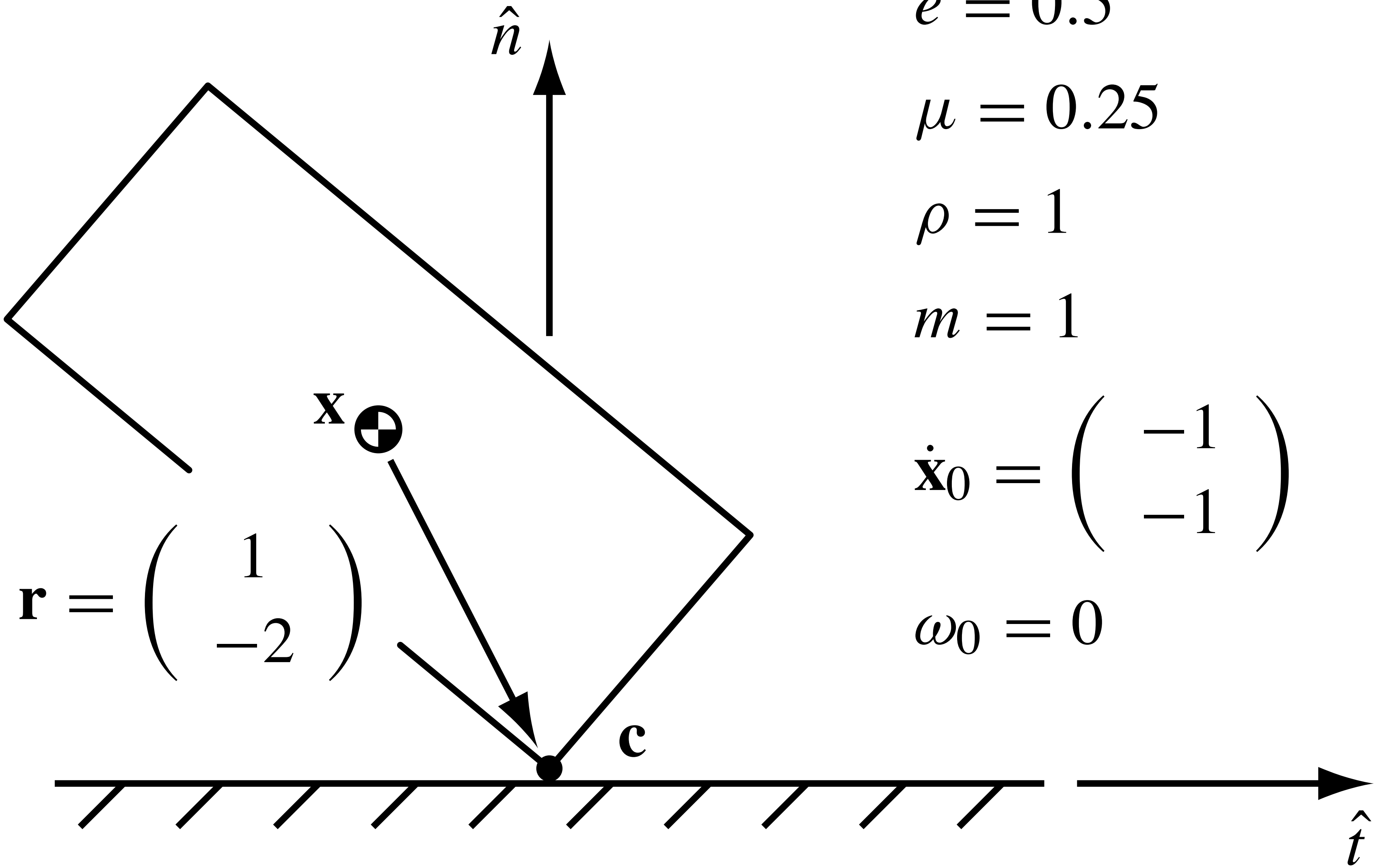
The sticking line is defined by the equation $\dot{c}_t = 0$,

$$\dot{c}_{0t} + \frac{\rho^2 + r_n^2}{\rho^2 m} I_t - \frac{r_t r_n}{\rho^2 m} I_n = 0 \tag{9}$$

and the compression line is defined the equation $\dot{c}_n = 0$,

$$\dot{c}_{0n} - \frac{r_t r_n}{\rho^2 m} I_t + \frac{\rho^2 + r_t^2}{\rho^2 m} I_n = 0 \tag{10}$$

Rigid body example



$$e = 0.5$$

$$\mu = 0.25$$

$$\rho = 1$$

$$m = 1$$

$$\dot{\mathbf{x}}_0 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$\omega_0 = 0$$

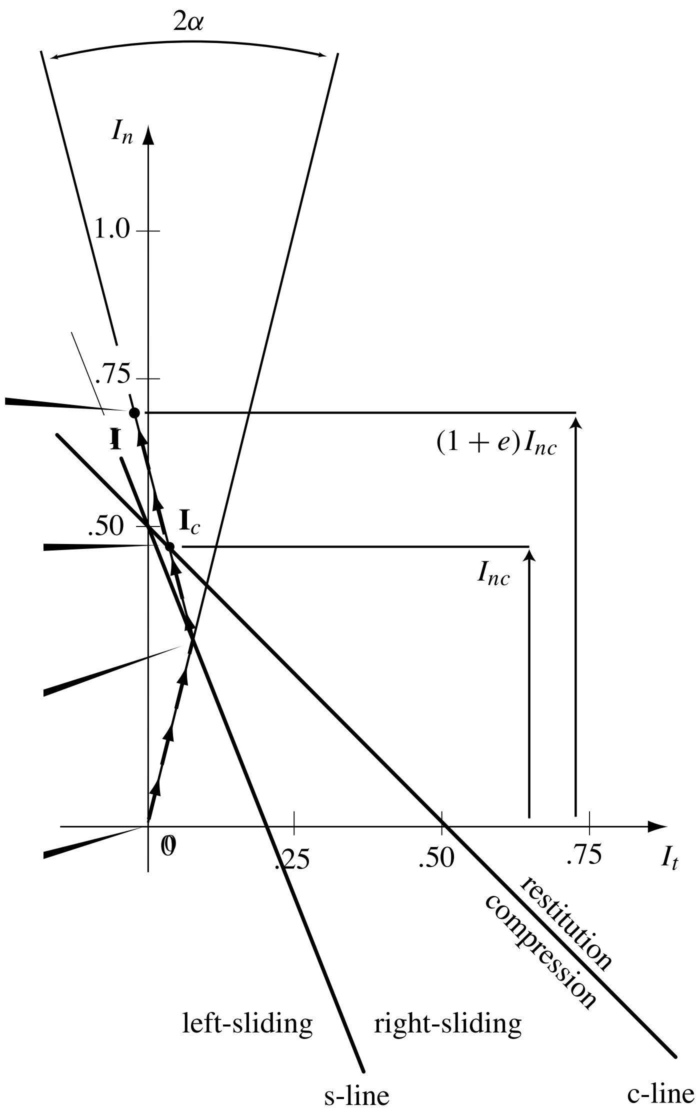
Analysis of rigid body example

Terminate at $I_n = (1+e)I_{nc}$.

At the c-line.
Maximum compression.
Switch to restitution.

At the s-line.
Switch to right-sliding.

Commence in compression,
left-sliding.



Sliding rod problem

$e = 0.5$

$\mu = \tan(30^\circ)$

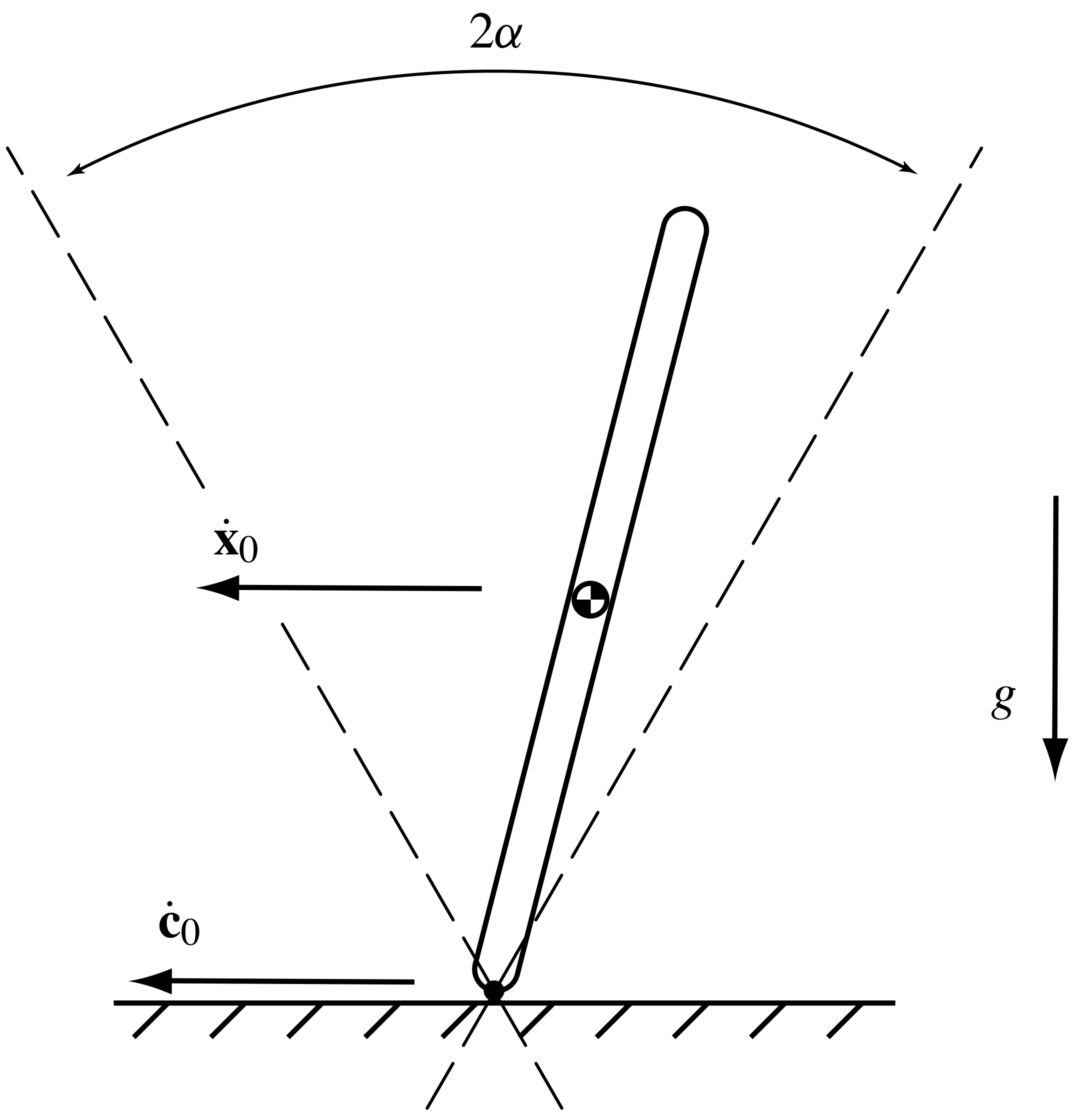
$\rho = 1$

$m = 1$

$\dot{\mathbf{x}}_0 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$\omega_0 = 0$

$\mathbf{r} = \begin{pmatrix} -1 \\ -4 \end{pmatrix}$



Sliding rod solution