6. Representing Rotation

*Mechanics of Manipulation*

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Outline.

- Generalities
- Axis-angle
- Rodrigues’s formula
- Rotation matrices
- Euler angles
Why representing rotations is hard.

- Rotations do not commute.
- The topology of spatial rotations does not permit a smooth embedding in Euclidean three space.
Choices

- More than three numbers
  - Rotation matrices
  - Unit quaternions. (aka Euler parameters)
- Many-to-one
  - Axis times angle (matrix exponential)
- Unsmooth and many-to-one
  - Euler angles
- Unsmooth and many-to-one and more than three numbers
  - Axis-angle
**Axis-angle**

Recall Euler’s theorem: every spatial rotation leaves a line of fixed points: the rotation axis.

Let $O, \hat{n}, \theta$, be …

Let $\text{rot}(\hat{n}, \theta)$ be the corresponding rotation.

Many to one:

$$\text{rot}(-\hat{n}, -\theta) = \text{rot}(\hat{n}, \theta)$$

$$\text{rot}(\hat{n}, \theta + 2k\pi) = \text{rot}(\hat{n}, \theta), \text{ for any integer } k.$$  

When $\theta = 0$, the rotation axis is indeterminate, giving an infinity-to-one mapping.
Representation

What do we want from a representation? For a start:

- Rotate points;
  Rodrigues’s formula
- Compose rotations;
  Using axis-angle? Ugh.
- (Convert to other representations.)
Rodrigues’s formula

Others derive Rodrigues’s formula using rotation matrices, missing the geometrical aspects.

Given point $\mathbf{x}$, decompose into components parallel and perpendicular to the rotation axis

$$
\mathbf{x} = \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{x}) - \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{x})
$$

Only $\mathbf{x}_\perp$ is affected by the rotation, yielding Rodrigues’s formula:

$$
\mathbf{x}' = \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{x}) + \sin \theta (\hat{\mathbf{n}} \times \mathbf{x}) - \cos \theta \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{x})
$$

A common variation:

$$
\mathbf{x}' = \mathbf{x} + (\sin \theta) \hat{\mathbf{n}} \times \mathbf{x} + (1 - \cos \theta) \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{x})
$$
Rotation matrices

Choose $O$ on rotation axis. Choose frame $(\hat{u}_1, \hat{u}_2, \hat{u}_3)$.
Let $(\hat{u}'_1, \hat{u}'_2, \hat{u}'_3)$ be the image of that frame.
Write the $\hat{u}'_i$ vectors in $\hat{u}_i$ coordinates, and collect them in a matrix:

\[
\begin{align*}
\hat{u}'_1 &= \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} = \begin{pmatrix} \hat{u}_1 \cdot \hat{u}'_1 \\ \hat{u}_2 \cdot \hat{u}'_1 \\ \hat{u}_3 \cdot \hat{u}'_1 \end{pmatrix} \\
\hat{u}'_2 &= \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} = \begin{pmatrix} \hat{u}_1 \cdot \hat{u}'_2 \\ \hat{u}_2 \cdot \hat{u}'_2 \\ \hat{u}_3 \cdot \hat{u}'_2 \end{pmatrix} \\
\hat{u}'_3 &= \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} = \begin{pmatrix} \hat{u}_1 \cdot \hat{u}'_3 \\ \hat{u}_2 \cdot \hat{u}'_3 \\ \hat{u}_3 \cdot \hat{u}'_3 \end{pmatrix}
\end{align*}
\]
So many numbers

A rotation matrix has nine numbers, but spatial rotations have only three degrees of freedom, leaving six excess numbers . . .

There are six constraints that hold among the nine numbers.

\[ |\mathbf{u}'_1| = |\mathbf{u}'_2| = |\mathbf{u}'_3| = 1 \]
\[ \mathbf{u}'_3 = \mathbf{u}'_1 \times \mathbf{u}'_2 \]

i.e. the \( \mathbf{u}'_i \) are unit vectors forming a right-handed coordinate system.

Such matrices are called orthonormal or rotation matrices.
Rotating a point

Let \((x_1, x_2, x_3)\) be coordinates of \(x\) in frame \((\hat{u}_1, \hat{u}_2, \hat{u}_3)\).

Then \(x'\) is given by the same coordinates taken in the \((\hat{u}_1', \hat{u}_2', \hat{u}_3')\) frame:

\[
x' = x_1\hat{u}_1' + x_2\hat{u}_2' + x_3\hat{u}_3'
\]

\[
= x_1 A\hat{u}_1 + x_2 A\hat{u}_2 + x_3 A\hat{u}_3
\]

\[
= A(x_1\hat{u}_1 + x_2\hat{u}_2 + x_3\hat{u}_3)
\]

\[
= Ax
\]

So rotating a point is implemented by ordinary matrix multiplication.
Rotating a point

Let $A$ and $B$ be coordinate frames. Notation:

- $\mathbf{x}$ a point
- $\mathbf{x}$ a geometrical vector, directed from an origin $O$ to the point $\mathbf{x}$; or, a vector of three numbers, representing $\mathbf{x}$ in an unspecified frame
- $\mathbf{A}\mathbf{x}$ a vector of three numbers, representing $\mathbf{x}$ in the $A$ frame

Let $\mathbf{BR}_A$ be the rotation matrix that rotates frame $B$ to frame $A$.

Then (see previous slide) $\mathbf{BR}_A$ represents the rotation of the point $\mathbf{x}$:

$$\mathbf{B}\mathbf{x}' = \mathbf{BR}_A \mathbf{B}\mathbf{x}$$

Note superscripts all match. Both points, and xform, must be written in same coordinate frame.
Coordinate transform

There is another use for $^{B}_A R$:

$^A x$ and $^B x$ represent the same point, in frames $A$ and $B$ resp.

To transform from $A$ to $B$:

$^B x = ^B_R A x$

For coord xform, matrix subscript and vector superscript “cancel”.

Rotation from $B$ to $A$ is the same as coordinate transform from $A$ to $B$. 
Example rotation matrix

\[ B^A_R = \begin{pmatrix} \begin{pmatrix} Bx_A \\ By_A \\ Bz_A \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \]

How to remember what \( B^A_R \) does? Pick a coordinate axis and see. The \( x \) axis isn't very interesting, so try \( y \):

\[ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \]
Nice things about rotation matrices

- Composition of rotations: \( \{R_1; R_2\} = R_2 R_1 \).
  \( \{x; y\} \) means do \( x \) then do \( y \).
- Inverse of rotation matrix is its transpose \( B R^{-1} = A R = B R^T \).
- Coordinate xform of a rotation matrix:

\[
BR = \begin{pmatrix} A & R \\ B & R \end{pmatrix}
\]
Converting $\text{rot}(\hat{n}, \theta)$ to $R$

Ugly way: define frame with $\hat{z}$ aligned with $\hat{n}$, use coordinate xform of previous slide.

Keen way: Rodrigues's formula!

$$x' = x + (\sin \theta) \hat{n} \times x + (1 - \cos \theta) \hat{n} \times (\hat{n} \times x)$$

Define “cross product matrix” $N$:

$$N = \begin{pmatrix}
0 & -n_3 & n_2 \\
n_3 & 0 & -n_1 \\
-n_2 & n_1 & 0
\end{pmatrix}$$

so that

$$Nx = \hat{n} \times x$$
using Rodrigues’s formula...

Substituting the cross product matrix $N$ into Rodrigues’s formula:

$$x' = x + (\sin \theta)Nx + (1 - \cos \theta)N^2x$$

Factoring out $x$

$$R = I + (\sin \theta)N + (1 - \cos \theta)N^2$$

That’s it! Rodrigues’s formula in matrix form. If you want to you could expand it:

$$
\begin{pmatrix}
  n_1^2 + (1 - n_1^2)c\theta & n_1n_2(1 - c\theta) - n_3s\theta & n_1n_3(1 - c\theta) + n_2s\theta \\
  n_1n_2(1 - c\theta) + n_3s\theta & n_2^2 + (1 - n_2^2)c\theta & n_2n_3(1 - c\theta) - n_1s\theta \\
  n_1n_3(1 - c\theta) - n_2s\theta & n_2n_3(1 - c\theta) + n_1s\theta & n_3^2 + (1 - n_3^2)c\theta
\end{pmatrix}
$$

where $c\theta = \cos \theta$ and $s\theta = \sin \theta$. Ugly.
Rodrigues’s formula for differential rotations

Consider Rodrigues’s formula for a differential rotation \( \text{rot}(\mathbf{n}, d\theta) \).

\[
x' = (I + \sin d\theta N + (1 - \cos d\theta) N^2)\mathbf{x}
\]

\[
= (I + d\theta N)\mathbf{x}
\]

so

\[
d\mathbf{x} = N\mathbf{x} \, d\theta
\]

\[
= \mathbf{n} \times \mathbf{x} \, d\theta
\]

It follows easily that differential rotations are vectors: you can scale them and add them up. We adopt the convention of representing angular velocity by the unit vector \( \mathbf{n} \) times the angular velocity.
Converting from $\mathbb{R}$ to $\text{rot}(\hat{n}, \theta)$ . . .

Problem: $\hat{n}$ isn’t defined for $\theta = 0$.

We will do it indirectly. Convert $\mathbb{R}$ to a unit quaternion (next lecture), then to axis-angle.
Euler angles

Three numbers to describe spatial rotations. \( ZY Z \) convention:

\[
(\alpha, \beta, \gamma) \mapsto \text{rot}(\gamma, \hat{z}'') \text{rot}(\beta, \hat{y}') \text{rot}(\alpha, \hat{z})
\]

Can we represent an arbitrary rotation?

- Rotate \( \alpha \) about \( \hat{z} \) until \( \hat{y}' \perp \hat{z}''' \);
- Rotate \( \beta \) about \( \hat{y}' \) until \( \hat{z}'' \parallel \hat{z}''' \);
- Rotate \( \gamma \) about \( \hat{z}'' \) until \( \hat{y}'' = \hat{y}''' \).

Note two choices for \( \hat{y}' \) . . .

. . . except sometimes infinite choices.
From \((\alpha, \beta, \gamma)\) to \(R\)

Expand \(\text{rot}(\alpha, \hat{z})\ \text{rot}(\beta, \hat{y})\ \text{rot}(\gamma, \hat{z})\)

(Why is that the right order?)

\[
\begin{pmatrix}
\cos \alpha & -\sin \alpha & 0 \\
\sin \alpha & \cos \alpha & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\cos \beta & 0 & \sin \beta \\
0 & 1 & 0 \\
-\sin \beta & 0 & \cos \beta
\end{pmatrix}
\begin{pmatrix}
\cos \gamma & -\sin \gamma & 0 \\
\sin \gamma & \cos \gamma & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\cos \alpha \cos \beta \cos \gamma - \sin \alpha \sin \gamma & -\cos \alpha \cos \beta \sin \gamma - \sin \alpha \cos \gamma & \cos \alpha \sin \beta \\
\sin \alpha \cos \beta \cos \gamma + \cos \alpha \sin \gamma & -\sin \alpha \cos \beta \sin \gamma + \cos \alpha \cos \gamma & \sin \alpha \sin \beta \\
-\sin \beta \cos \gamma & \cos \beta \sin \gamma & \cos \beta
\end{pmatrix}
\]

\(1\)
From $R$ to $(\alpha, \beta, \gamma)$ the ugly way

Case 1: $r_{33} = 1$, $\beta = \pi$. $\alpha - \gamma$ is indeterminate.

$$R = \begin{pmatrix}
\cos(\alpha + \gamma) & -\sin(\alpha + \gamma) & 0 \\
\sin(\alpha + \gamma) & \cos(\alpha + \gamma) & 0 \\
0 & 0 & 1
\end{pmatrix}$$

Case 2: $r_{33} = -1$, $\beta = -\pi$. $\alpha + \gamma$ is indeterminate.

$$R = \begin{pmatrix}
-\cos(\alpha - \gamma) & -\sin(\alpha - \gamma) & 0 \\
-\sin(\alpha - \gamma) & \cos(\alpha - \gamma) & 0 \\
0 & 0 & 1
\end{pmatrix}$$

For generic case: solve 3rd column for $\beta$. (Sign is free choice.) Solve third column for $\alpha$ and third row for $\gamma$.

... but there are numerical issues ...
From $R$ to $(\alpha, \beta, \gamma)$ the clean way

Let

\[ \sigma = \alpha + \gamma \]
\[ \delta = \alpha - \gamma \]

Then

\[ r_{22} + r_{11} = \cos \sigma (1 + \cos \beta) \]
\[ r_{22} - r_{11} = \cos \delta (1 - \cos \beta) \]
\[ r_{21} + r_{12} = \sin \delta (1 - \cos \beta) \]
\[ r_{21} - r_{12} = \sin \sigma (1 + \cos \beta) \]

(No special cases for $\cos \beta = \pm 1$?)

Solve for $\sigma$ and $\delta$, then for $\alpha$ and $\gamma$, then finally

\[ \beta = \tan^{-1}(r_{13} \cos \alpha + r_{23} \sin \alpha, r_{33}) \]